Presentation

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REAL LINE TOPOLOGY

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Sets

A well defined collection of objects is called a set. The sets are denoted by capital Letters A, B, C, ...

Example

1. $\mathbb{N} = \{1, 2, 3, 4, ...\}$. *This is called the set of Natural numbers.*
2. $\mathbb{W} = \{0, 1, 2, 3, ...\}$. *This is called the set of whole numbers.*
3. $\mathbb{Z} = \{... -3, -2, -1, 0, 1, 2, 3, ...\}$. *This is called the set of all integers.*
4. *The set* $\mathbb{Q} = \{\frac{p}{q} : p, q \in Z \text{ and } q \neq 0\}$ *is called the set of all rational numbers.*

5. The set $\mathbb{Q}^{\star}$ of all those numbers which when expressed in decimal form are expressible neither interminating decimal nor in repeating decimals, is called the set of irrational numbers.
6. The totality of rationals and irrationals forms the set $\mathbb{R}$ of all real numbers.
7. $X = \{a, b, c\}$.

Finite and Infinite sets

A set in which the process of counting of elements surely comes to an end is called a finite set.

A set which is not finite is called an infinite set.

Example

1. *In the above examples, 7 is called finite.*
2. *In the above examples, 1 to 6 are called infinite.*

Empty set

A set consisting of no element at all is called an empty set and it is denoted by ϕ.

Subsets

Let A and B be two sets such that each element of A is also contained in B. Then A is called a subset of set B and we write, $A \subseteq B$.

Example

1. $\mathbb{N} \subseteq \mathbb{W} \subseteq \mathbb{Z}$.
2. $\mathbb{Q} \subseteq \mathbb{R}$.
3. $\mathbb{Q}^\star \subseteq \mathbb{R}$.
4. *The empty set is a subset of every set.*

Union of sets

The union of two sets A and B denoted by $A \cup B$ is the set consisting of all those elements each one of which is contained either in A or in B or in both A and B.
Thus, $A \cup B = \{x : x \in A \text{ or } x \in B\}$.

Example

1. $\mathbb{N} \cup \mathbb{W} = \mathbb{W}$.
2. $\mathbb{Q} \cup \mathbb{Q}^\star = \mathbb{R}$.

Intersection of sets

The intersection of two sets A and B denoted by $A \cap B$ is the set consisting of all those elements which are common to both A and B.
Thus $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

Example

1. $\mathbb{Q} \cap \mathbb{Q}^{\star} = \phi$.
2. $\mathbb{R} \cap \mathbb{Q} = \mathbb{Q}$.
3. $\mathbb{R} \cap \mathbb{Q}^{\star} = \mathbb{Q}^{\star}$.

Ordered pair

An ordered pair is a pair of entries made in a specify order, separated by a comma and enclosed within parentheses.

Example

$(1, 2)$

Topology

Let X be a non-empty set. A class $\mathcal{T}$ of subsets of X is a topology on X iff $\mathcal{T}$ satisfies the following axioms.

1. X and ϕ belong to $\mathcal{T}$.
2. The union of any number of sets in $\mathcal{T}$ belongs to $\mathcal{T}$.
3. The intersection of any two sets in $\mathcal{T}$ belongs to $\mathcal{T}$.

The members of $\mathcal{T}$ are called open sets and the pair (X, $\mathcal{T}$) is called a topological space.

Example

1. $\mathcal{T} = \{\phi, \mathbb{R}, \mathbb{Q}\}$.
2. $\mathcal{T} = \{\phi, \mathbb{R}, \mathbb{Q}^{\star}\}$.
3. $\mathcal{T} = \{\phi, \mathbb{R}, \mathbb{Q}, \mathbb{Q}^{\star}\}$.
4. $\mathcal{T} = \{\phi, \mathbb{R}, \mathbb{N}, \mathbb{Q}\}$.

Types of functions

1. **Onto function or surjection:** A function $f: A \to B$ is called a surjection or an onto function if each element in B has a pre-image in A. (or) $f(A) = B$.
2. **One-one function or injection:** A function $f: A \to B$ is called an injection or a one-one function if distinct elements in A have distinct images in B. i.e. an element in B has at most one pre-image in A. Symbolically the function f is one-one if $f(x) = f(x') \Rightarrow x = x'$ (x and $x' \in A$).
3. **One-one and onto function or bijection:** A function $f: A \to B$ which is both one-one and onto is called a bijection.
4. **Identity function:** The function $f: A \to A$ defined as $f(x) = x$, $\forall\, x \in A$ is called the identity function on A.
   Obviously the identity function is a bijection.

Example

Let $A = \{a, b, c\}$ *and* $B = \{x, y, z, w\}$. *Define* $f : A \to B$ *by* $f(a) = z$ *and* $f(c) = x$. *Then f is not a function.*

Example

Let $A = \{a, b, c\}$ *and* $B = \{x, y, z, w\}$. *Define* $f : A \to B$ *by* $f(a) = y$; $f(b) = x$ *and* $f(c) = w$. *Then f is a one-one function.*

Example

Let $A = \{1, 3, 5\}$ *and* $B = \{3, 9, 15\}$. *Define* $f : A \to B$ *by* $f(x) = 3x$ *for all* $x \in A$. *Then f is a one-one onto function.*

Definition

Two sets A and B are said to be equivalent if there exists a bijection f from A to B.

Definition

A set A is said to be countably infinite if A is equivalent to the set $\mathbb{N}$ of natural numbers.

Example

1. *Since $\mathbb{N}$ is equivalent to $\mathbb{N}$, the set $\mathbb{N}$ of natural numbers is countable.*
2. $\{2, 4, 6, \ldots, 2n, \ldots\}$ *is countably infinite.*
3. *The set $\mathbb{Z}$ of integers is countably infinite.*
4. *The set* $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\}$ *is countably infinite.*

Definition

A set A is said to be countable if it is finite or countably infinite.

Example

1. $\{2, 4, 6, ..., 2n, ...\}$ is countable.
2. The set $\mathbb{Z}$ of integers is countable.
3. The set $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ...\}$ is countable.
4. The set $\mathbb{Q}^+$ of positive rational numbers is countable.
5. The set $\mathbb{Q}$ of rational numbers is countable.
6. The set $\{0, 1, \frac{1}{2}, \frac{1}{3}, ...\}$ is countable.

Definition

A set which is not countable is called uncountable.

Example

1. *(0, 1] is uncountable.*
2. *The set $\mathbb{R}$ of real numbers is uncountable.*
3. *The set $\mathbb{Q}^\star$ of irrational numbers is uncountable.*
4. *Any interval in $\mathbb{R}$ which contains more than one point is uncountable.*

Definition

Let A be a subset of a space X, a point p in X is called a condensation point of A if for each open set U containing p, $U \cap A$ is uncountable.

Note: A countable subset A has no condensation points.

τ_u denotes the usual topology on $\mathbb{R}$.

Example

1. *In $(\mathbb{R}, \tau_u)$, $\mathbb{N}$, the set of natural numbers, have no condensation points.*
2. *In $(\mathbb{R}, \tau_u)$, $\mathbb{Q}$, the set of rational numbers, have no condensation points.*

Definition

A subset A is called ω-open if for each $x \in A$, there exists an open set U containing x such that $U - A$ is countable.

Note: In (X, τ), if A is open, then A is ω-open.

Example

The set $\mathbb{Q}^{\star}$ of irrational numbers in the real line space with the usual topology is ω-open but not open.