1 SET THEORY

1.1. INTRODUCTION TO SETS

In this chapter, we shall deal with the most fundamental items in mathematics–The Sets. Sets were first explained by mathematician George Cantor. The application of set theory play a very important role in economics and business decisions.

Definition

A set is a collection of well-defined and well-distinguished objects.

Note :
- (*i*) **The 'objects' in a set are called the members or elements of the set.**
- (*ii*) **The words 'well-defined' imply that there is a definite rule or condition which tell us whether a particular object (element) belongs to the set or not.**
- (*iii*) **The words 'well-distinguished' imply that all the objects (elements) of a set must be different. In other words, the elements of a set must not repeat themselve.**
- (*iv*) **The sets are usually denoted by the capital letters of English alphabet say. A, B, C, D ..... X, Y, Z etc.**

Illustration 1.

1. The collection of members in a particular family is a set.
2. The collection of cities in Punjab is a set.
3. The collection of Books in the college library is a set.
4. The collection of vowels of English alphabet is a set.

1.1.1 Some Standard Sets

N = The set of all natural numbers. W = The set of whole numbers.
I or Z = The set of integers Q = The set of rational numbers.
R = The set of real numbers. C = The set of complex numbers.

For a given set A and given object a, one of the following statements is true.

(*i*) 'a' is in the collection A.
(*ii*) 'a' is not in the collection A.

In symbols, If a is an object of A, then it is written as $a \in A$ and read as "belong to A".
If a is not in the collection A, then it is written as $a \notin A$ and is read as "does not belong to A".
e.g. let $A = \{a, b, c, d\}$ then $a \in A, b \in A, d \in A$ but $1 \notin A, 2 \notin A, e \notin A, f \notin A$

Illustration 2. ***Which of the following are sets.***

(*i*) ***A collection of most talented persons of India.*** (*ii*) ***A collection of all even integers.***
(*iii*) ***A collection of beautiful girls of the world.*** (*iv*) ***A collection of rivers in India.***

Solution.

(*i*) The collection of most talented persons of India is not a set, since the term 'most talented person' is vague and is not well-defined.
(*ii*) The collection of even integers is a set.
(*iii*) The collection of beautiful girls of the world is not a set, since the term 'beautiful' is not well-defined.
(*iv*) The collection of rivers of India is a set.

1.1.2 List of Symbols used in Set Theory

$\in$ = 'belongs to' $\notin$ = 'does not belong to'
ϕ = 'empty set' $\neq$ = 'not equal to'
$\subseteq$ = 'is a subset of' $\supseteq$ = 'Contains or equal to'
$\subset$ = 'is contained in' $\rightarrow$ = 'Mapped to' or 'goes to'
$\Rightarrow$ = 'implies that' $\Leftrightarrow$ = 'if and only if'

: = Such that
$\cap$ = ~~Union~~ Intersection
$\nless$ = 'not less than'
$\cup$ = Union
$\ngtr$ = 'not greater than'
$\forall$ = 'for all values of'

1.2. REPRESENTATION OF A SET

There are two methods to represent a set :

(*i*) **Roster or Tabulation method** (*ii*) **Set builder form.**

(*i*) **Roster or Tabulation method.** In this method, all the elements of a set are listed (without repetition) by separating the elements by commas and enclosing them in curly brackets { }. When the set contain an infinite number of elements, then it is not possible to list all the elements. In this case, we list a few elements of set, followed by dots (.....) within the curly brackets.

e.g. $A = \{2, 4, 6, 8, 10\}$, $B = \{1, 3, 5, 7, ...\}$

(*ii*) **Set builder form.** In this method, the set is represented by specifying the common property of the elements. In other words, in set-builder form, we state a property, to which every element must possess in order to be a member of the set.

In the builder notation a set is described as $\{x : x$ has property P $\}$ and is read as 'The set of elements x such that x has the property P'. The colon : or/ is read as 'such that'.

Illustration 3.

(*i*) $A = \{a, e, i, o, u\}$ (Tabular Form)
$A = \{x : x$ is a vowel in English alphabet$\}$ (Set builder form)
(*ii*) $B = \{2, 4, 6, 8, 10\}$ (Tabular form)
$B = \{x : x$ is a positive even integer $\leq 10\}$ (Set Builder form)
(*iii*) $C = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$ (Tabular form)
$C = \{x : x$ is a factor of 36 $\}$ (Set Builder form)

Illustration 4. ***Write the following sets in Roster form.***

(*i*) $A = \{x : x$ ***is a letter in the word 'MATHEMATICS'*** $\}$
(*ii*) $B = \{x : x$ ***is a prime number which is a divisor of 60*** $\}$
(*iii*) $C = \{x : x$ ***is an integer and*** $-2 < x < 8\}$
(*iv*) $D = \{x : x$ ***is a two digit number such that sum of its digits is 9***$\}$

Solution.

(*i*) $A = \{M, A, T, H, E, I, C, S\}$ (*ii*) $B = \{2, 3, 5\}$
(*iii*) $C = \{-1, 0, 1, 2, 3, 4, 5, 6, 7\}$ (*iv*) $D = \{18, 27, 36, 45, 54, 63, 72, 81, 90\}$

Illustration 5. ***Express the following sets by using set-builder methods.***

(*i*) $A = \{5, 10, 15, ...\}$ (*ii*) $B = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}\right\}$
(*iii*) $C = \{0, 1, 4, 9, 16, ...\}$ (*iv*) $D = \{a, e, i, o, u\}$

Solution. (*i*) $A = \{x : x$ is a natural number and multiple of 5$\}$
(*ii*) $B = \left\{x : x = \frac{n}{n+1}; n \in N \text{ and } 1 \leq n \leq 7\right\}$ (*iii*) $C = \{x : x = n^2 : n \in z\}$
(*iv*) $D = \{x : x$ is vowel of English alphabet $\}$

1.3. TYPES OF SETS

1.3.1. Finite Set

A set is said to be finite if it has finite number of elements *i.e.* the process of counting the different elements of the set terminate after a finite number of steps.

i.e. (*i*) $A = \{1, 2, 3, 4\}$ (*ii*) $B = \{x : x$ is a day in the week $\}$ (*iii*) $C = \{a, e, i, o, u\}$

The number of elements in a set is called cardinal number and is denoted as n (A)

Hence $n(A) = 4$, $n(B) = 7$, $n(C) = 5$.

1.3.2. Infinite Set

A set is said to be infinite if it has infinite number of elements. Here the process of counting all the different elements of the set never terminate.

e.g. (*i*) N = { x : x is a natural number } (*ii*) P = { x : x is a prime number }
(*iii*) R = { x : x is a real number }

Illustration 6. ***Which of the following sets are finite and which are infinite***

(*i*) $A = \{x : x \text{ is a positive even integer} \leq 10\}$
(*ii*) $B = \{x : x \text{ is a positive integer greater than } 50\}$
(*iii*) $C = \{x : x \in N \text{ and } 2x = 6\}$
(*iv*) $D = \{x : x \text{ is a month of the year}\}$

Solution. (*i*) Here, A = { 0, 2, 4, 6, 8, 10 } which is a finite.
(*ii*) B = {51, 52, 53, 54, ... } which is an infinite set.
(*iii*) C = {3} which is a finite set.
(*iv*) D = {January, February, March, April, May, June, July, August, September, October, November, December} which is a finite set.

1.3.3. Empty Set

A set which has no element is called null set or empty set.
An empty set is denoted by the symbol ϕ.

e.g. (*i*) A = { x : x is a positive integer < 1 }
(*ii*) B = { x : x is both even and odd }
(*iii*) C = { x : x is a prime number between 24 and 28 }
(*iv*) D = { $x : x \in$ N and $2x - 3 = 0$ }

Obviously each of above sets has no element, therefore they are empty sets.

Illustration 7. ***Which of the following sets are empty sets.***

(*i*) $A = \{x : x \text{ is an odd natural number divisible by } 2\}$
(*ii*) $B = \{x : x \text{ is a point of intersection of any two parallel lines}\}$

Solution. (*i*) As no odd natural number is divisible by 2 $\therefore$ A = ϕ.
(*ii*) As any two parallel lines never intersect $\therefore$ B = ϕ.

1.3.4. Singleton set or Unit Set

A set which contain only one element is called singleton set or unit set.

e.g. (*i*) A = { 2 } (*ii*) B = $\{x : x > 4 \text{ and } x < 6\}$
(*iii*) C = { x : x is a ruling Prime minister of India}

1.3.5. Equivalent or Similar Set

Two sets A and B are said to be equivalent or similar if they have same number of elements *i.e.* n (A) = n (B). Similar sets are denoted as A ~ B and read as A is similar to B.

e.g. If A = { 1,2,3 } and B = { a, b, c }
Then, n(A) = n(B) = 3 $\Rightarrow$ A ~ B

1.3.6. Equal Sets

Two sets A and B are said to be equal if they have exactly the same elements. It is denoted as A = B *e.g.* If A = {1, 2, 3, 4}
and B = { $x : n \in$ N and $x \leq 4$ }. Then A = B.

Illustraion 8. ***Which of the following statements are true :***

(*i*) *If* $A = \{x : x^2 = 4, x \in N\}$; $B = \{-2\}$ *then* $A \neq B$.
(*ii*) *If* $A = \{1, 2, 3, 4, 5, 5\}$; $B = \{2, 1, 3, 5, 4\}$, *then* $A = B$.

Solution. (*i*) Here $A = \{x : x^2 = 4 : x \in N\} = \{x : x = \pm\sqrt{4} : x \in N\} = \{2\}$
But $B = \{-2\}$ $\Rightarrow A \neq B$ $\Rightarrow$ (*i*) is true.

(*ii*) Here A = { 1, 2, 3, 4, 5 }, also B = { 1, 2, 3, 4, 5 } $\Rightarrow$ A = B $\Rightarrow$ (*ii*) is true.

1.3.7. Disjoint Sets

Two sets A and B are said to be disjoint if they do not have any common element.
e.g. If A = { 1, 2, 3 } and B = { a, b, c } then A and B are disjoint sets.

1.3.8. Subset

Let A and B be two sets, then A is said to be a subset of B if all the elements of A are present in B. In symbols, it is denoted as $A \subseteq B$ and is read 'A is a subset of B' or 'A is contained in B'.

e.g. If A = {1, 2, 3} and B = {1, 2, 3, 4, 5 } then $A \subseteq B$.

> **Note :** (*i*) $A \subseteq B$ then if $x \in A \Rightarrow x \in B$.
> (*ii*) If $A \subseteq B$, then we also say that 'B is a superset of A' and denoted as $B \supseteq A$.
> (*iii*) Every set is a subset of itself *i.e.* $A \subseteq A$.
> (*iv*) Empty set 'ϕ' is a subset of every set *i.e.* $\phi \subseteq A$.

1.3.9. Comparable Sets

Two sets A and B are said to be comparable iff either $A \subset B$ or $B \subset A$.

e.g. Let A = { 2, 3, 5 }, B = { 2, 3, 5, 6 } C = { 1, 5}

As $A \subseteq B \Rightarrow$ A and B are comparable sets and $A \not\subset C$ and $C \not\subset A \Rightarrow$ A and C are non-comparable sets.

1.3.10. Proper subset

A set A is said to be a proper subset of B, if A is a subset of B and A is not equal to B. In other words, A is a proper subset of B if $A \subset B$ but $A \neq B$. *i.e.* there is at least one element of B which is not in A.

If a subset of a set is not proper, then it is called improper subset.

ϕ and A are improper subsets of A.

1.3.11. Power Set

The collection of all the possible subsets of a given finite set A is called the power set of A and it is denoted by P(A).

P(A) = { X : X is a subset of A}

e.g. If A = { 1, 2, 3 }

then P(A) = {ϕ, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}}

> **Note :** If a set A has *n* elements thsen P(A) has 2^n elements, *i.e.* the total numbers of subsets of a finite set A of order *n* is 2^n.

1.3.12. Universal Set

A set which contain all the elements of different sets under consideration or in a given problem is called the universal set. It is usually denoted as U or X. The universal set is not a fixed set. It varies from situation to situation.

e.g. (*i*) Let A = { 1, 2, 3 }, B = {a, b, c} and C = {p, q, r}

Then U = { 1, 2, 3, a, b, c, p, q, r}

(*ii*) If A = {All male students of Punjab}

B = {All the students of girls colleges in Punjab}

C = {All the students getting scholarship in Punjab}

The parent set or universal set of all these sets is the set of all the students in Punjab.

SOLVED EXAMPLES

Example 1. *Check which of the following are sets.*

(*i*) *A = {x : x is a member of football team of a college}*

(*ii*) *B = {x : x is a woman president of India}*

(*iii*) *C = {x : x is intelligent persons of India}*

Solution. (*i*) A is a set. (*ii*) B is a set

(*iii*) C is not a set because the term intelligent persons is not well defined.

Example 2. *Write the following sets in set-builder form :*

(*i*) *A = {1, 8, 27, 64}*

(*ii*) *B = { 7, 14, 21, 28, ... 63}*

(*iii*) *C = {1, 3, 5, 7, 9, ... 17}*

(*iv*) *D = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}*

Solution. (*i*) A = $\{x : x = n^3, 0 < n < 5\}$

(ii) $B = \{x : x \text{ is a positive multiple of 7 and } x \leq 63\}$
(iii) $C = \{x : x \text{ is an odd natural number} \leq 17\}$
(iv) $D = \{x : x \text{ is whole number} \leq 9\}$

Example 3. ***Write the following sets in Tabular form.***
(i) $A = \{x : x \text{ is a positive integer}\}$
(ii) $B = \{x : x \text{ is a positive multiple of 6 and is} < 65\}$
(iii) $C = \{x : x \text{ is a square of integer from 1 to 5}\}$
(iv) $D = \{x : x^2 - 7x - 12 = 0\}$

Solution. (i) $A = \{1, 2, 3, 4, 5, ...\}$ (ii) $B = \{6, 12, 18, 24, 30, ... 60\}$
(iii) $C = \{1, 4, 9, 16, 25\}$ (iv) $D = \{3, 4\}$

Example 4. ***Which of the following sets is finite or infinite.***
(i) $A = \{x : x \text{ is a human hair on the body}\}$
(ii) $B = \{x : x \text{ is a citizen of India}\}$
(iii) $C = \{x : x \text{ is a coin in India}\}$
(iv) $D = \{x : x \text{ a student of Govt. College for Women, Ludhiana}\}$
(v) $E = \{x : x \text{ is a natural number}\}$

Solution. (i) A is an Infinite set (ii) B is a finite set (iii) C is a finite set
(iv) D is a finite set (v) E is an infinite set.

Example 5. ***Which of the following are true.***
(i) $\phi = \{0\} = \{\phi\}$
(ii) $\{x : x \text{ is a letter of the word 'PARE'}\} = \{x : x \text{ is a letter of the word 'PEAR}\}$'
(iii) $\{a, b, c\} \sim \{\alpha, \beta, \gamma\}$ (iv) $\{x : x^2 - 5x + 6 = 0\} = \{2, 3\}$

Solution.
(i) False $\because \phi$ denotes empty set which contain no element whereas $\{0\}$ and $\{\phi\}$ are unit sets which contain 0 and ϕ respectively.
(ii) True $\because \{x : x$ is a letter of the word 'PARE'$\} = \{P, A, R, E\}$ and $\{x : x$ is a letter of the word 'PEAR'$\} = \{P, E, A, R\} \Rightarrow \{P, A, R, E\} = \{P, E, A, R\}$
(iii) True.
$\because$ No. of elements is $\{a, b, c\} = 3$, No. of elements is $\{\alpha, \beta, \gamma\} = 3 \Rightarrow \{a, b, c\} \sim \{\alpha, \beta, \gamma\}$
(iv) True $\because \{x : x^2 - 5x + 6 = 0\} = \{x : (x-2)(x-3) = 0\} = \{x : x = 2 \text{ or } 3\} = \{2, 3\}$

Example 6. ***Check whether the following sets are null set or unit sets.***
(i) $A = \{x : x \text{ is a three legged animal}\}$
(ii) $B = \{x : x \text{ is a magnet with one pole}\}$
(iii) $C = \{x : x \text{ is an Indian living on the moon}\}$
(iv) $D = \{x : x \text{ is a president of India}\}$.

Solution. (i) $A = \phi$, empty set. (ii) $B = \phi$, empty set.
(iii) $C = \phi$, empty set. (iv) D is a unit set.

Example 7. ***If $A = \{2, 4, 6\}$, $B = \{1, 2, 6\}$, $C = \{4, 2, 6\}$ and $D = \{2, 6\}$, then which of the following are true.***
(i) $A = B$ (ii) $A \subset B$ (iii) $A = C$ (iv) $D \subset A$
(v) $D \subset C$ (vi) $4 \in A$ (vii) $3 \in D$ (viii) $\phi \subset A$

Solution. (i) False (ii) False (iii) True (iv) True
(v) True (vi) True (vii) False (viii) True.

EXERCISE–1

1. Which of the following are sets.
   (i) The collection of intelligent students in India.
   (ii) The collection of positive multiples of 5.
   (iii) The collection of rich persons in Chandigarh.
2. Write the following sets in Roster form.
   (i) $A = \{x : x \in M \text{ and } x^2 = 25\}$

(ii) $B = \{x : x \text{ is a positive multiple of 3 and } x < 27\}$

(iii) $C = \{x : x \in M \text{ and } x \text{ is a prime number between 6 and 30}\}$

3. Describe the following sets in set-builder form.

   (i) $A = \{6, 12, 18, 24\}$ (ii) $B = \{1, 4, 9, 16, 25\}$

   (iii) $C = \left\{\frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \frac{1}{12}\right\}$

4. Which of the following are null sets :

   (i) $A = \{x : x < 1 \text{ and } x > 3\}$ (ii) $B = \{x : x \text{ is an even prime number}\}$

   (iii) $C = \{x : x + 4 = 4\}$

5. Which of the following are singleton sets.

   (i) $A = \{x : 2x - 1 = 0 : x \in Q\}$ (ii) $B = \{x : x^3 - 1 = 0, x \in R\}$

   (iii) $C = \{x : x^2 = 4 : x \in R\}$

6. Which of the following are finite sets.

   (i) $A = \{x : x \text{ is a tree in India}\}$

   (ii) $B = \{x : x \text{ is an Indian who has got Nobel prize}\}$

   (iii) $C = \{x : x \text{ is a person in age group 15-30 year in India}\}$

   (iv) $D = \{x : x \text{ is a star in the sky}\}$

7. Which of the following are correct.

   (i) $\{1, 2, 3\} = \{5, 6, 7\}$

   (ii) $\{x : x \text{ is a divisor of } 12\} = \{1, 2, 3, 4, 6, 12\}$

   (iii) If $A = \{2, 4, 6\}$, $B = \{1, 3, 5, 7, ...\}$ $C = \{10, 20, 30, ...\}$ then $U = \{x : x \text{ is a positive integer}\}$

   (iv) If $A = \{x : x \text{ is a living female prime minister of India}\}$ then $A = \phi$

   (v) $A = \{a, b, c, d\}$, $B = \{a, e, b, c, d, u\}$ are non-comparable sets.

   (vi) If $X = \{1, 2, 3\}$, $Y = \{1\}$ then $Y \subseteq X$

   (vii) Power set of $\{1, 2, 3, 4\}$ is $\{\phi, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\} \{1, 3, 4\} \{1, 4, 2\} \{2, 4, 5\} \{1, 2, 3, 4\}\}$

8. List down all the subsets of $\{a, b, c\}$

9. Write down all the proper subsets of $\{1, 2, 3\}$

10. Find the power set of $A = \{1, \{2\}\}$

ANSWERS

1. (ii)

2. (i) $A = \{5\}$ (ii) $B = \{3, 6, 9, 12, 15, 18, 21, 24\}$ (iii) $C = \{7, 11, 13, 17, 19, 23, 29\}$

3. (i) $A = \{x : x \text{ is a positive multiple of } 6\}$ (ii) $B = \{x : x \text{ is a perfect square} \leq 25\}$

   (iii) $C = \{x : x = \frac{1}{2n} : n \in N : 2 \leq n \leq 6\}$

4. (i), 5. (i), (ii) 6. (i) finite (ii) finite (iii) finite (iv) infinite

7. (i) correct (ii) correct (iii) correct (iv) correct (v) correct (vi) correct (vii) correct

8. $\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}$ $\{a, b, c\}$

9. $\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}$ 10. $P(A) = \{\phi, \{1\}, \{\{2\}\}, \{1, \{2\}\}$

1.4. VENN DIAGRAM

Venn-diagrams named after John Venn (1834-1883) a famous mathematician are the diagrammatical representation of sets and their properties.

Venn-diagrams give a clear and better understanding of set theory and its applications.

In Venn-diagrams, the universal set is usually represent the interior of rectangles while any other set is represented by the interior of a circle or a simple closed curve.

The adjoining figure display the venn-diagram for $A \subset B$.

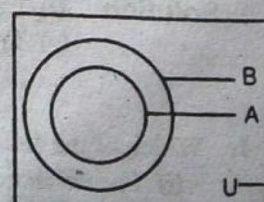
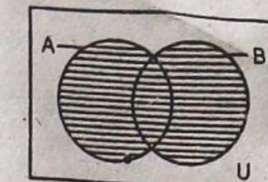


Fig 1.

1.5. OPERATIONS ON SETS

1.5.1 Union of two sets



AUB (Shaded Area)

Fig 2

The union of two sets A and B written as $A \cup B$ is a new set consisting of all the elements of A and B.

$\therefore A \cup B = \{x : x \in A \text{ or } x \in B\}$

Illustration 9. ***If A = set of alphabets in 'PUNJAB' and B = Set of alphabets in 'BOMBAY', Find A ∪ B.***

Solution. Now $A = \{P, U, N, J, A, B\}$,

$B = \{B, O, M, B, A, Y\}$

Hence $A \cup B = \{P, U, N, J, A, B, O, M, Y\}$

1.5.1.1 Properties of Union of the sets

(*i*) $A \cup A = A$

e.g. $A = \{a, b, c\} \Rightarrow A \cup A = \{a, b, c\} \Rightarrow A \cup A = A$ [Idempotent law]

(*ii*) $A \cup B = B \cup A$

e.g. $A = \{a, b, c\}$

$B = \{c, d, f\}$ $\therefore A \cup B = \{a, b, c, d, f\}$

and $B \cup A = \{a, b, c, d, f\}$ $\Rightarrow A \cup B = B \cup A$ [Commutative law]

(*iii*) $A \cup \phi = A$ $A = \{a, b, c\}$

$A \cup \phi = \{a, b, c\}$ and $\because \phi = \{\ \}$ $\Rightarrow A \cup \phi = A$ [Identity law]

(*iv*) $A \cup U = U$ (U is universal set)

e.g. $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $A = \{2, 4, 6\}$

$A \cup U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ $\Rightarrow A \cup U = U$

(*v*) $A \cup A^C = U$

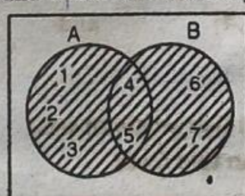
e.g. $A = \{2, 4, 6\}$; $A^C = \{1, 3, 5, 7, 8, 9\}$

$A \cup A^C = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ $\Rightarrow A \cup A^C = U$

Illustration 10. ***If A = {1, 2, 3, 4, 5}, B = {4, 5, 6, 7}. Find A ∪ B and draw venn diagram.***

Solution. $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$

Clearly 4, 5 are common elements and therefore taken only once. See Fig. 3

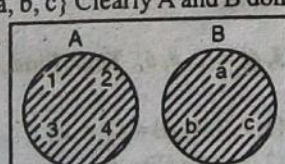


AUB (Shaded Area)

Fig 3.

Illustration 11. ***If A = {1, 2, 3, 4} B = {a, b, c}. Find A ∪ B and draw venn's diagram.***

Solution. $A \cup B = \{1, 2, 3, 4, a, b, c\}$ Clearly A and B don't overlap. See Fig. 4



AUB – Shaded Area

Fig 4.

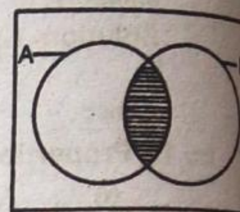
Illustration 12. ***If A = {1, 2, 3, 4, 5, 6} and B = {2, 4, 6}. Find A ∪ B and draw venn's diagram.***

Solution. Clearly $B \subseteq A$ $\Rightarrow A \cup B = \{1, 2, 3, 4, 5, 6\} = A$. See Fig. 5

Fig 5.

1.5.2 Intersection of the sets

If A and B be two sets, then intersection of A and B denoted as $A \cap B$ is a set of all those elements which belong to both A and B. See Fig. 6.



A∩B (Shaded Area)

Fig 6.

i.e. $A \cap B \{x : x \in A \text{ and } x \in B\}$

e.g. Let $A = \{2, 4, 6, 8\}$ and $B = \{4, 8, 12, 10\}$

then $A \cap B = \{4, 8\}$

1.5.2.1 Properties of Intersection of two sets

(*i*) $A \cap A = A$

e.g. $A = \{1,2,3\}$

$A \cap A = \{1,2,3\} = A$ $\qquad A \cap A = A$ (Idempotent law)

(*ii*) $A \cap B = B \cap A$

e.g. $A = \{1,2,3,4\}$ $\qquad B = \{2,4,6\}$

$\Rightarrow A \cap B = \{2,4\}$ and $B \cap A = \{2,4\}$

$\Rightarrow A \cap B = B \cap A$ (Commutative law)

(*iii*) $A \cap \phi = \phi$

e.g. $A = \{a, b, c\}$ $\qquad \phi = \{\ \}$

$\Rightarrow A \cap \phi = \{\ \} = \phi \Rightarrow A \cap \phi = \phi$

(*iv*) $A \cap U = A$

e.g. $U = \{1,2,3,4,5,6,7,8,9\}$ $\qquad A = \{2,3,5,7\}$

$\Rightarrow A \cap U = \{2,3,5,7\} \Rightarrow A \cap U = A$ (Identity Law)

(*v*) $A \cap A^C = \phi$

e.g. $A = \{1,2,3,4\}$ and $A^C = \{5,6,7,8,9\}$

(Where $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$)

$A \cap A^C = \phi$

(*vi*) If $A \cap B = \phi$ then A and B are disjoint sets.

1.5.2.2 Intersection of More than two Sets

The intersection of more than two sets is obtained by collecting all the member (elements) which are common in all the sets. See Fig. 7.

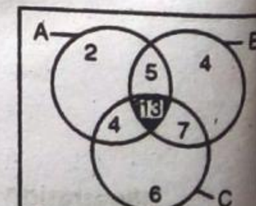


Fig 7.

e.g. Let $A = \{1,2,3,4,5\}$

$B = \{1,3,5,7,9\}$

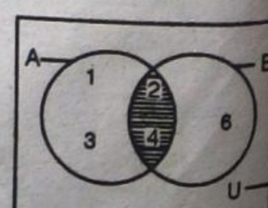
$C = \{1,3,4,6,7\}$

$A \cap B \cap C = \{1,3\}$

Illustration 13. ***If $A = \{1, 2, 3, 4\} = \{2, 4, 6\}$. Then Find $A \cap B$ and draw venn diagram.***

Solution. Here $A = \{1,2,3,4\}, B = \{2,4,6\}$

$\Rightarrow A \cap B = \{2,4\}$. See Fig. 8.



A∩B → Shaded Area.

Fig 8.

Illustration 14. *If $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$.* **Find $A \cap B$ also draw venn diagram.**

Solution. Here $A = \{1,2,3,4\}$

and $B = \{a, b, c\}$

if $A \cap B = \{\ \}$ $\because$ there is no element

Common to both A and B = $A \cap B$ is a null set

$A \cap B = \phi$. See Fig. 9.

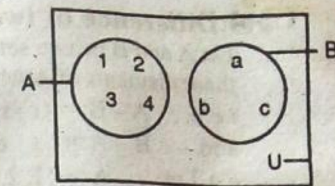


Fig 9.

Illustration 15. *If $A = \{1, 2, 3, 4, 5, 6\}$ $B = \{2, 4, 6\}$ then. Find $A \cap B$ also draw venn diagram.*

Solution. Here $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{2, 4, 6\}$

$\Rightarrow$ $A \cap B = \{2, 4, 6\} = B$

$\because$ B is a proper subset of A.

$\therefore$ $A \cap B = B$. See Fig. 10.

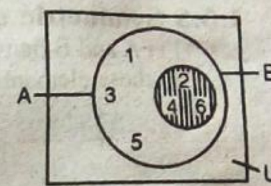


Fig 10.

1.5.3 Complement of a Set

Let U be universal set and A be any subset of U. Then complement of A is the set of all those elements of U, which are not in A.
It is denoted by A′ or A^C.

$\therefore$ $A^C = \{x : x \notin A \text{ and } x \in U\}$

> **Note : If the universal set is changed, the complement of A also changes.**

Illustration 16. *If (i) A = {All female in Punjab} and U = {All persons in Punjab} then find A^C. (ii) U = {All female in India}, then find A^C and also draw venn diagram.*

Solution. (*i*) A^C = {All persons in Punjab except females}

(*ii*) If U = {All female in India} then A^C

= {All female in India, except those in Punjab}

$A^C \longrightarrow$ Shaded Area

Fig 11.

1.5.3.1 Properties of Complement of a Set

(*i*) Any set A divides the universal set U into two parts A and A^C.
Therefore, if an element x does not belong to A then it must belong to A^C. In symbols, $x \notin A \Rightarrow x \in A^C$

The converse is also true. *i.e.* $x \notin A^C\ \forall\ x \in A$.

(*ii*) Which further imply that the intersection of a set A and its complement is null set (empty set).
i.e. $A \cap A^C = \phi$

(*iii*) The union of a set A and its complement is the universal set *i.e.* $A \cup A^C = U$

(*iv*) If A is a proper subset of B, then complement of B is a proper subset of complement of A.
$A \subset B \Rightarrow B^C \subset A^C$.

(*v*) $(A^C)^C = A$

e.g. $U = \{1,2,3,4,5,6,7,8,9\}$ $A = \{2,4,6,8\}$

$A^C = \{1,3,5,7,9\}$ $(A^C)^C = \{2,4,6,8\}$

Clearly $(A^C)^C = A$

(*vi*) $U^C = \phi$

e.g. $U = \{1,2,3,4,5,6,7,8,9\}$ $U^C = \{x : x \notin U\}$

$\Rightarrow$ $\{\ \} = \phi$ $\Rightarrow$ $U^C = \phi$.

(*vii*) $\phi^C = U$

e.g. $U = \{1,2,3,4,5,6,7,8,9\}$ $\phi^C = \{x : x \notin \phi\}$

$= \{1,2,3,4,5,6,7,8,9\}$ $\Rightarrow$ $\phi^C = U$

1.5.4 Difference of two sets

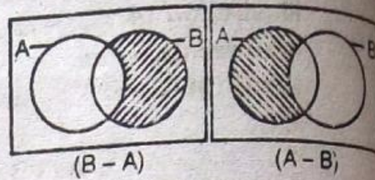


Fig 12.

Let A and B be two sets, then difference $A-B$ is the set of all those elements of A which are not in B.

i.e. $A-B=\{x : x \in A, x \notin B\}$

and $B-A=\{x : x \in B, x \notin A\}$

e.g. Let $A=\{1,2,3\}, B=\{2,4,6,8\}$

then $A-B=\{1,3\}$ and $B-A=\{4,6,8\}$

1.5.5 Symmetric difference of two sets.

Let A and B be two sets, then the symmetric difference of A and B denoted as $A \Delta B$ is then set of all those elements which are in $A \cup B$ but not in $A \cap B$.

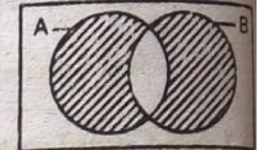


Fig 13.

i.e. $A \Delta B=\{x : x \in A \cup B \text{ and } x \notin A \cap B\}$

or $A \Delta B=\{x : x \in (A-B) \cup (B-A)\}$

$A \Delta B=(A \cup B)-(A \cap B)$

e.g. Let $A=\{1,2,3,\} B=\{3,4,5,6\}$

$A-B=\{1,2\}$; $B-A=\{4,5,6\}$

$(A-B) \cup (B-A)=\{1,2,4,5,6\}$

Also $A \cup B=\{1,2,3,4,5,6\}, A \cap B=\{3\}$

$(A \cup B)-(A \cap B)=\{1,2,3,4,5,6\}$

$\Rightarrow$ $A \Delta B=(A-B) \cup (B-A)=(A \cup B)-(A \cap B)=\{1,2,4,5,6\}$

1.6. LAWS OF SETS

1.6.1 Associative Laws

If A, B and C are any three sets, then

(*i*) $(A \cup B) \cup C=A \cup (B \cup C)$ (*ii*) $A \cap (B \cap C)=(A \cap B) \cap C$

Proof. (*i*) Let x be any arbitrary element of the set $(A \cup B) \cup C$.

Then

$x \in (A \cup B) \cup C \Rightarrow x \in (A \cup B)$ or $x \in C$

$\Rightarrow$ $(x \in A$ or $x \in B)$ or $x \in C$ $\Rightarrow$ $x \in A$ or $(x \in B$ or $x \in C)$

$\Rightarrow$ $x \in A$ or $x \in (B \cup C)$ $\Rightarrow$ $x \in A \cup (B \cup C)$

$\therefore$ $(A \cup B) \cup C \subseteq A \cup (B \cup C)$ ...(1)

Let y be any arbitrary element of the set $A \cup (B \cup C)$. Then

$y \in A \cup (B \cup C) \Rightarrow y \in A$ or $y \in (B \cup C) \Rightarrow$ $y \in A$ or $(y \in B$ or $y \in C)$

$\Rightarrow$ $(y \in A$ or $y \in B)$ or $y \in C$ $\Rightarrow$ $y \in (A \cup B)$ or $y \in C$

$\Rightarrow$ $y \in (A \cup B) \cup C$

$\therefore$ $A \cup (B \cup C) \subseteq (A \cup B) \cup C.$ ...(2)

From (1) and (2), we have

$$\mathbf{(A \cup B) \cup C=A \cup (B \cup C).}$$

(*ii*) Let x be any arbitrary element of the set $A \cap (B \cap C)$. Then

$x \in A \cap (B \cap C) \Rightarrow x \in A$ and $x \in (B \cap C)$

$\Rightarrow$ $x \in A$ and $(x \in B$ and $x \in C)$ $\Rightarrow$ $(x \in A$ and $x \in B)$ and $x \in C$

$\Rightarrow$ $x \in (A \cap B)$ and $x \in C$ $\Rightarrow$ $x \in (A \cap B) \cap C$

$\therefore$ $A \cap (B \cap C) \subseteq (A \cap B) \cap C$ ...(1)

Again let y be any arbitrary element of the set $(A \cap B) \cap C$. Then

$y \in (A \cap B) \cap C \Rightarrow y \in (A \cap B)$ and $y \in C$

$\Rightarrow$ $(y \in A$ and $y \in B)$ and $y \in C$ $\Rightarrow$ $y \in A$ and $(y \in B$ and $y \in C)$

$\Rightarrow$ $y \in A$ and $y \in (B \cap C)$ $\Rightarrow$ $y \in A \cap (B \cap C)$

$\therefore$ $(A \cap B) \cap C \subseteq A \cap (B \cap C)$ ...(2)

From (1) and (2), we have

$$\mathbf{A \cap (B \cap C)=(A \cap B) \cap C.}$$

.6.2. Distributive Laws

If A, B and C are any three sets, then

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cup C)$

Proof. (i) Let x be any arbitrary element of the set $A \cup (B \cap C)$.

Then

$$x \in A \cup (B \cap C) \Rightarrow x \in A \text{ or } x \in (B \cap C) \Rightarrow x \in A \text{ or } (x \in B \text{ and } x \in C)$$

$$\Rightarrow (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C)$$

$\Rightarrow$ $x \in A \cup B$ and $x \in A \cup C$ $\Rightarrow$ $x \in (A \cup B) \cap (A \cup C)$

$\therefore$ $A \cup (B \cap C) \subseteq (A \cap B) \cap (A \cup C)$ ...(1)

Again let y be any arbitrary element of the set $(A \cup B) \cap (A \cup C)$. Then

$$y \in (A \cup B) \cup (A \cup C) \Rightarrow y \in (A \cup B) \text{ and } y \in (A \cup C)$$

$\Rightarrow$ $(y \in A \text{ or } y \in B)$ and $(y \in A \text{ or } y \in C)$ $\Rightarrow$ $y \in A$ or $(y \in B \text{ and } y \in C)$

$\Rightarrow$ $y \in A$ or $y \in (B \cap C)$ $\Rightarrow$ $y \in A \cup (B \cap C)$

$\therefore$ $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$ ...(2)

From (1) and (2), we have

$$\mathbf{A \cup (B \cap C) = (A \cup B) \cap (A \cup C).}$$

(ii) Let x be any arbitrary element of the set $A \cap (B \cup C)$. Then

$$x \in A \cap (B \cup C) \Rightarrow x \in A \text{ and } x \in (B \cup C)$$

$\Rightarrow$ $x \in A$ and $(x \in B \text{ or } x \in C)$ $\Rightarrow$ $(x \in A \text{ and } x \in B)$ or $(x \in A \text{ and } x \in C)$

$\Rightarrow$ $x \in (A \cap B)$ or $x \in (A \cap C)$ $\Rightarrow$ $x \in (A \cap B) \cup (A \cap C)$

$\therefore$ $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ ...(1)

Again let y be any arbitrary element of the set $(A \cap B) \cup (A \cap C)$. Then

$y \in (A \cap B) \cup (A \cap C)$ $\Rightarrow$ $y \in (A \cap B)$ or $y \in (A \cap C)$

$\Rightarrow$ $(y \in A \text{ and } y \in B)$ or $(y \in A \text{ and } y \in C)$ $\Rightarrow$ $y \in A$ and $(y \in B \text{ or } y \in C)$

$\Rightarrow$ $y \in A$ and $y \in (B \cup C)$ $\Rightarrow$ $y \in A \cap (B \cup C)$

$\therefore$ $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ ...(2)

From (1) and (2), we have $\mathbf{A \cap (B \cup C) = (A \cap B) \cup (A \cap C).}$

1.6.3. De-Morgan's Laws.

If A and B are any two sets, then (i) $(A \cup B)^C = A^C \cap B^C$ (ii) $(A \cap B)^C = A^C \cup B^C$

Proof. (i) Let x be any arbitrary element of set $(A \cup B)^C$. Then

$x \in (A \cup B)^C$ $\Rightarrow$ $x \notin A \cup B$ $\Rightarrow$ $x \notin A$ and $x \notin B$

$\Rightarrow$ $x \in A^C$ and $x \in B^C$ $\Rightarrow$ $x \in A^C \cap B^C$

$\therefore$ $(A \cup B)^C \subseteq A^C \cap B^C$ ...(1)

Again let y be any arbitrary element of the set $A^C \cap B^C$. Then

$y \in A^C \cap B^C$ $\Rightarrow$ $y \in A^C$ and $y \in B^C \Rightarrow$ $y \notin A$ and $y \notin B$

$\Rightarrow$ $y \notin A \cup B$ $\Rightarrow$ $y \in (A \cup B)^C$

$\therefore$ $A^C \cap B^C \subseteq (A \cup B)^C$ ...(2)

From (1) and (2), we have

$$\mathbf{(A \cup B)^C = A^C \cap B^C}$$

(ii) Let x be any arbitrary elements of the set $(A \cap B)^C$. Then

$x \in (A \cap B)^C$ $\Rightarrow$ $x \notin (A \cap B)$ $\Rightarrow$ $x \notin A$ or $x \notin B$

$\Rightarrow$ $x \in A^C$ or $x \in B^C$ $\Rightarrow$ $x \in A^C \cup B^C$

$\therefore$ $(A \cap B)^C \subseteq A^C \cup B^C$ ...(1)

Again, let y be any arbitary element of the set $A^C \cup B^C$. Then

$y \in A^C \cup B^C$ $\Rightarrow$ $y \in A^C$ or $y \in B^C$ $\Rightarrow$ $y \notin A$ or $y \notin B$

$\Rightarrow$ $y \notin (A \cap B)$ $\Rightarrow$ $y \in (A \cap B)^C$

$\therefore$ $A^C \cup B^C \subseteq (A \cap B)^C$ ...(2)

From (1) and (2), we have

$$(A \cap B)^C = A^C \cup B^C$$

1.6.4. De-Morgan's Law on Difference of Sets

If A, B and C be any three sets, then

(*i*) $A-(B\cup C)=(A-B)\cap(A-C)$ (*ii*) $A-(B\cap C)=(A-B)\cup(A-C)$

Proof. (*i*) Let x be any arbitrary element of the set $A-(B\cup C)$.

Then $x\in A-(B\cup C)\Rightarrow x\in A$ and $x\notin(B\cup C)$

$\Rightarrow$ $x\in A$ and ($x\notin B$ and $x\notin C$) $\Rightarrow$ ($x\in A$ and $x\notin B$) and ($x\in A$ and $x\notin C$)

$\Rightarrow$ $x\in(A-B)$ and $x\in(A-C)$ $\Rightarrow$ $x\in[(A-B)\cap(A-C)]$

$\therefore$ $A-(B\cup C)\subseteq(A-B)\cap(A-C)$

Again let y be any arbitrary element of the set $(A-B)\cap(A-C)$.

Then

$y\in(A-B)\cap(A-C)$ $\Rightarrow$ $y\in(A-B)$ and $y\in(A-C)$

$\Rightarrow$ ($y\in A$ and $y\notin B$) and ($y\in A$ and $y\notin C$) $\Rightarrow$ $y\in A$ and ($y\notin B$ and $y\notin C$)

$\Rightarrow$ $y\in A$ and $y\notin(B\cup C)$ $\Rightarrow$ $y\in A-(B\cup C)$

$\therefore$ $(A-B)\cap(A-C)\subseteq A-(B\cup C)$ ...(2)

From (1) and (2), we have

$$A-(B\cup C)=(A-B)\cap(A-C)$$

(*ii*) Let x be any arbitrary element of the set $A-(B\cap C)$. Then

$x\in A-(B\cap C)\Rightarrow x\in A$ and $x\notin(B\cap C)$

$\Rightarrow$ $x\in A$ and ($x\notin B$ or $x\notin C$) $\Rightarrow$ ($x\in A$ and $x\notin B$) or ($x\in A$ and $x\notin C$)

$\Rightarrow$ $x\in(A-B)$ or $x\in(A-C)$ $\Rightarrow$ $x\in(A-B)\cup(A-C)$

$\therefore$ $A-(B\cap C)\subseteq(A-B)\cup(A-C)$ ...(1)

Again let y be any element of the set $(A-B)\cup(A-C)$. Then

$y\in(A-B)\cup(A-C)\Rightarrow y\in(A-B)$ or $y\in(A-C)$

$\Rightarrow$ ($y\in A$ and $y\notin B$) or ($y\in A$ and $y\notin C$) $\Rightarrow$ $y\in A$ and ($y\notin B$ or $y\notin C$)

$\Rightarrow$ $y\in A$ and $y\notin(B\cap C)$ $\Rightarrow$ $y\in A-(B\cap C)$

$\therefore$ $(A-B)\cup(A-C)\subseteq A-(B\cap C)$ ...(2)

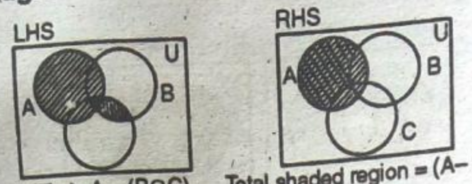
From (1) and (2), we have

$$A-(B\cap C)=(A-B)\cup(A-C)$$

1.6.5 Some more Laws of Sets

1. **Idempotent laws:** (*i*) $A\cup A=A$ (*ii*) $A\cap A=A$
2. **Identity laws:** (*i*) $A\cup\phi=A$ (*ii*) $A\cap U=A$
3. **Commutative laws:** (*i*) $A\cup B=B\cup A$ (*ii*) $A\cap B=B\cap A$
4. **Laws of complement:** (*i*) $U^C=\phi$ (*ii*) $\phi^C=U$ (*iii*) $(A^C)^C=A$
 (*iv*) $A\cup A^C=U$ (*v*) $A\cap A^C=\phi$

Proof by Venn Diagram



▨ → A − (B∩C)   Total shaded region = (A− B) ∪ (A−C)

Fig 17.

1.8. PRACTICAL APPLICATIONS OF SETS

We shall now discuss the practical applications of operations of set thoery.

1.8.1 To find a formula for n (A∪B)

Let A and B be two sets. Then the following two cases arise

Case I. A and B are disjoint sets *i.e.*, A and B have no common element.

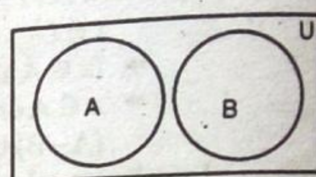


Fig 18.

From Venn-diagram Fig. 18 H is clear that

$\therefore\ n(A \cup B) = n(A) + n(B)$ ...(1)

> **Note.** **The number of elements in a set A is called cardinal number and is denoted by n (A).**

Case II. A and B are not disjoint sets.

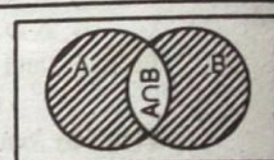


Fig 19.

(*i*) From Venn-diagram [Fig. (19)], it is clear that

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$ ...(2)

(*ii*) From Venn-diagram (Fig (20)), it is clear that

$A \cup B = (A - B) \cup (A \cap B) \cup (B - A)$

Now as (A − B), (A ∩ B), (B − A) are all disjoint sets

$\therefore$ By case I,

$n(A \cup B) = n(A - B) + n(A \cap B) + n(B - A)$ ...(3)

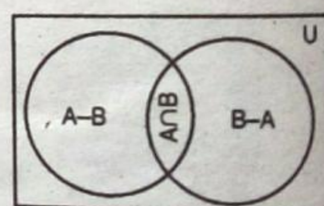


Fig 20.

(*iii*) From Venn-diagram [Fig. (20)], it is clear that

$A = (A - B) \cup (A \cap B)$

$\therefore\ n(A) = n(A - B) + n(A \cap B)$ ...(4)

Similarly $n(B) = n(B - A) + n(A \cap B)$ ...(5)

(*iv*) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$ ...(6)

1.8.2. Results on Cardinal Number of Some sets

If A, B and C are finite sets and U be the universal set, then we have following results :

(*i*) $n(A \cup B) = n(A) + n(B)$ if A and B are disjoint sets.

(*ii*) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(*iii*) $n(A \cup B) = n(A - B) + n(A \cap B) + n(B - A)$

(*iv*) $n(A) = n(A - B) + n(A \cap B)$

Cor. 1. $n(A - B) = n(A) - n(A \cap B)$

Cor. 2. $n(A - B) = n(A \cup B) - n(B)$

(*v*) $n(B) = n(B - A) + n(A \cap B)$

Cor. 1. $n(B - A) = n(B) - n(A \cap B)$

Cor. 2. $n(B - A) = n(A \cup B) - n(A)$

(*vi*) $n(A^c) = n(U) - n(A)$

(*vii*) $n(A^c \cap B^c) = n(A \cup B)^c = n(U) - n(A \cup B)$

(*viii*) $n(A^c \cup B^c) = n(A \cap B)^c = n(U) - n(A \cap B)$

(*ix*) $n(A \cap B^c) = n(A) - n(A \cap B)$

(x) $n(A \cap B) = n(A \cup B) - n(A \cup B^c) - n(A^c \cup B)$

(xi) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$

SOLVED EXAMPLES

Example 1. ***If A, B and C are three sets and U is the universal set such that $n(U) = 700$, $n(A) = 200$, $n(B) = 300$ and $n(A \cap B) = 100$. Find $n(A^c \cap B^c)$.***

Solution. We have $A^c \cap B^c = (A \cup B)^c$

$$\therefore \quad n(A^c \cap B^c) = n((A \cup B)^c)$$
$$= n(U) - n(A \cup B) = n(U) - [n(A) + n(B) - n(A \cap B)]$$
$$= 700 - (200 + 300 - 100) = 300.$$

Example 2. ***Let A and B be two sets such that : $n(A) = 20$, $n(A \cup B) = 42$ and $n(A \cap B) = 4$. Find (i) $n(B)$ (ii) $n(A - B)$ (iii) $n(B - A)$***

Solution. (i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\Rightarrow \quad 42 = 20 + n(B) - 4 \Rightarrow n(B) = 42 - 16 = 26$

(ii) $\quad n(A - B) = n(A) - n(A \cap B) = 20 - 4 = 16$

(iii) $\quad n(B - A) = n(B) - n(A \cap B) = 26 - 4 = 22.$

Example 3. ***A survey shows that 76% of the Indians like oranges, whereas 62% like bananas. What percentage of the Indians like both oranges and bananas?***

Solution. Let A and B denote the set of Indians who like oranges and bananas respectively.

Then $\quad n(A) = 76, n(B) = 62, n(A \cup B) = 100$

$\therefore \quad n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\Rightarrow \quad n(A \cap B) = n(A) + n(B) - n(A \cup B) = 76 + 62 - 100 = 38.$

$\therefore$ 38% of the Indians like both oranges and bananas.

Example 4. ***In a group of 50 persons, 14 drink tea but not coffee and 30 drink tea. Find: (i) how many drink tea and coffee both ; (ii) how many drink coffee but not tea.***

Solution. Let A and B be the set of persons who drink tea and coffee respectively.

Then $\quad n(A \cup B) = 50, n(A - B) = 14, n(A) = 30$

(i) $\quad n(A - B) = n(A) - n(A \cap B)$

$\Rightarrow \quad n(A \cap B) = n(A) - n(A - B) = 30 - 14 = 16$

(ii) Required number $= n(B - A) = n(B) - n(A \cap B)$

Now $\quad n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\Rightarrow \quad 50 = 30 + n(B) - 16 \quad \Rightarrow \quad n(B) = 50 - 14 = 36.$

So, $\quad n(B - A) = n(B) - n(A \cap B) \quad \Rightarrow n(B - A) = 36 - 16 = 20.$

Example 5. ***For a certain test, a candidate could offer English or Hindi or both the subjects. Total number of students was 500, of whom 350 appeared in English and 90 in both the subjects. Use set operations to show :***
(i) How many appeared in English only ? (ii) How many appeared in Hindi ?
(iii) How many appeared in Hindi only?

Solution. Let A be the set of candidates who take English and B be the set of candidates who take Hindi. Now $A \cap B$ = the set of candidates who take Hindi and English both

$A - B$ = the set of candidates who offer English only

$B - A$ = the set of candidates who offer Hindi only.

According to given conditions

$(A \cup B) = 500, n(A) = 350, n(A \cap B) = 90$

(i) $\quad n(A) = n(A - B) + n(A \cap B)$

$\Rightarrow \quad 350 = n(A - B) + 90 \quad \Rightarrow n(A - B) = 350 - 90 = 260.$

Hence the number of candidates who appeared in English only are 260.

(ii) $\quad n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\Rightarrow \quad 500 = 350 + n(B) - 90 \quad \Rightarrow \quad n(B) = 500 - 350 + 90 = 240$

Hence the number of candidates who appeared in Hindi are 240.

(iii) $\quad n(B) = n(B - A) + n(A \cap B)$

$\Rightarrow \quad 240 = n(B - A) + 90 \quad \Rightarrow \quad n(B - A) = 240 - 90 = 150.$

Hence number of candidates who appeared in Hindi only are 150.

Example 6. *(i) If X and Y are two sets such that $n(X) = 17, n(Y) = 23$ and $n(X \cup Y) = 38$, find $n(X \cap Y)$.*

(ii) If S and T are two sets such that S has 21 elements, T has 32 elements and $S \cap T$ has 11 elements, how many elements does $S \cup T$ have ?

(iii) If X and Y are two sets such that X has 40 elements, $X \cup Y$ has 60 elements and $X \cap Y$ has 10 elements, how many elements does Y have ?

(iv) In a group of 65 people, 40 like cricket, 10 like both cricket and tennis. How many like tennis only and not cricket ? How many like tennis ?

Solution. *(i)* $n(X \cup Y) = n(X) + n(Y) - n(X \cap Y)$

$\Rightarrow \quad 38 = 17 + 23 - n(X \cap Y) \quad \Rightarrow \quad n(X \cap Y) = 17 + 23 - 38 = 2$

(ii) $n(S \cup T) = n(S) + n(T) - n(S \cap T) = 21 + 32 - 11 = 53 - 11 = 42$

(iii) $n(Y) = n(X \cup Y) - n(X) + n(X \cap Y) = 60 - 40 + 10 = 30$

(iv) $n(T) = n(C \cup T) - n(C) + n(C \cap T) = 65 - 40 + 10 = 35$

$\therefore$ Number of people who like tennis only

$n(T - C) = n(T) - n(C \cap T) = 35 - 10 = 25$

$\therefore$ Number of persons liking Tennis only and not cricket = 25

and Number of persons liking Tennis = 35

Example 7. *(i) In a group of people, 50 speak both English and Hindi and 30 people speak English but not Hindi. All the people speak at least one of the two language. How many people speak English ?*

(ii) In a survey of 100 persons it was found that 28 read magazine A, 30 read magazine B, 42 read magazine C, 8 read magazines A and B, 10 real magazines A and C, 5 read magazines B and C and 3 read all the three magazines. Find :

(a) How many read none of the three magazines ?

(b) How many read magazine C only ?

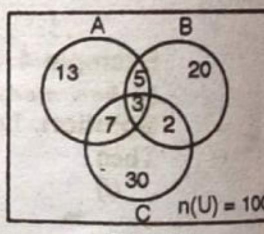


Fig 21.

Solution. *(i)* $n(E) = n(E - H) + n(E \cap H)$

$= 30 + 50 = 80$

(ii) Given, $n(U) = 100, n(A) = 28$

$n(B) = 30, \quad n(C) = 42$

$n(A \cap B) = 8, \quad n(A \cap C) = 10$

$n(B \cap C) = 5, \quad n(A \cap B \cap C) = 3$

$n(A) + n(B) + n(C) = 28 + 30 + 42 = 100$

(a) Number of persons who read none of the magazines

$= 100 - (13 + 5 + 20 + 7 + 3 + 2 + 30)$

$= 100 - 80 = 20$ [See Venn Diagram]

(b) Number of persons who read magazine C only = 30 [See Venn Diagram]

Example 8. *A survey of 500 television viewers produced the following information; 285 watch football, 195 watch hockey, 115 watch basketball, 45 watch football and basketball, 70 watch football and hockey, 50 watch hockey and basketball, 50 do not watch any of the three games. How many watch all the three games ? How many watch exactly one of the three games ?*

Solution. Let N = Total number of television viewers = 500

$n(F) = 285$, $n(H) = 195$, $n(B) = 115$, $n(F \cap B) = 45$, $n(F \cap H) = 70$, $n(H \cap B) = 50$, $n(F^c \cap H^c \cap B^c) = 50$, where F, H and B denote respectively the number of viewers who watch football, hockey and basketball respectively.

Now $\quad n(F^c \cap H^c \cap B^c) = 50$

$\Rightarrow \quad N - n(F \cup H \cup B) = 50 \quad \Rightarrow n[(F \cup H \cup B)^c] = 50$

$\Rightarrow 500 - [n(F) + n(H) + n(B) - n(F \cap H) - n(F \cap B) - n(H \cap B) + n(F \cap H \cap B)] = 50$

$\Rightarrow n(F \cap H \cap B) = 500 - 285 - 195 - 115 + 70 + 50 + 50 + 45 - 50 = 20$

(i) $\therefore$ Required number $= n(F \cap H \cap B) = 20$

(ii) Required number $= n(F \cap H^c \cap B^c) + n(F^c \cap H^c \cap B) + n(F^c \cap H \cap B^c)$

$= n(F) + n(H) + n(B) - 2[n(F \cap H) + n(H \cap B) + n(B \cap F)] + 3n(F \cap H \cap B)$

$= 285 + 195 + 115 - 2[170 + 50 + 45] + 3(20)$

$= 595 - 2[165] + 60 = 595 - 330 + 60 = 595 - 270 = 325.$

Indices

3.1 Indices

The lowest factors of 2000 are $2 \times 2 \times 2 \times 2 \times 5 \times 5 \times 5$. These factors are written as $2^4 \times 5^3$, where 2 and 5 are called **bases** and the numbers 4 and 3 are called **indices**.

When an index is an integer it is called a **power**. Thus, 2^4 is called 'two to the power of four', and has a base of 2 and an index of 4. Similarly, 5^3 is called 'five to the power of 3' and has a base of 5 and an index of 3.

Special names may be used when the indices are 2 and 3, these being called 'squared' and 'cubed', respectively. Thus 7^2 is called 'seven squared' and 9^3 is called 'nine cubed'. When no index is shown, the power is 1, i.e. 2 means 2^1.

Reciprocal

The **reciprocal** of a number is when the index is -1 and its value is given by 1 divided by the base. Thus the reciprocal of 2 is 2^{-1} and its value is $\frac{1}{2}$ or 0.5. Similarly, the reciprocal of 5 is 5^{-1} which means $\frac{1}{5}$ or 0.2

Square root

The **square root** of a number is when the index is $\frac{1}{2}$, and the square root of 2 is written as $2^{1/2}$ or $\sqrt{2}$. The value of a square root is the value of the base which when multiplied by itself gives the number. Since $3 \times 3 = 9$, then $\sqrt{9} = 3$. However, $(-3) \times (-3) = 9$, so $\sqrt{9} = -3$. There are always two answers when finding the square root of a number and this is shown by putting both a $+$ and a $-$ sign in front of the answer to a square root problem. Thus $\sqrt{9} = \pm 3$ and $4^{1/2} = \sqrt{4} = \pm 2$, and so on.

Laws of indices

When simplifying calculations involving indices, certain basic rules or laws can be applied, called the **laws of indices**. These are given below.

(i) When multiplying two or more numbers having the same base, the indices are added. Thus

$$3^2 \times 3^4 = 3^{2+4} = 3^6$$

(ii) When a number is divided by a number having the same base, the indices are subtracted. Thus

$$\frac{3^5}{3^2} = 3^{5-2} = 3^3$$

(iii) When a number which is raised to a power is raised to a further power, the indices are multiplied. Thus

$$(3^5)^2 = 3^{5 \times 2} = 3^{10}$$

(iv) When a number has an index of 0, its value is 1. Thus $3^0 = 1$

(v) A number raised to a negative power is the reciprocal of that number raised to a positive power. Thus $3^{-4} = \dfrac{1}{3^4}$

Similarly, $\dfrac{1}{2^{-3}} = 2^3$

(vi) When a number is raised to a fractional power the denominator of the fraction is the root of the number and the numerator is the power.

Thus $\quad 8^{2/3} = \sqrt[3]{8^2} = (2)^2 = 4$

and $\quad 25^{1/2} = \sqrt[2]{25^1} = \sqrt{25^1} = \pm 5$

(Note that $\sqrt{\ } \equiv \sqrt[2]{\ }$)

3.2 Worked problems on indices

Problem 1. Evaluate: (a) $5^2 \times 5^3$, (b) $3^2 \times 3^4 \times 3$ and (c) $2 \times 2^2 \times 2^5$

From law (i):

(a) $5^2 \times 5^3 = 5^{(2+3)} = 5^5 = 5 \times 5 \times 5 \times 5 \times 5 = \mathbf{3125}$

(b) $3^2 \times 3^4 \times 3 = 3^{(2+4+1)} = 3^7$

$= 3 \times 3 \times \ldots$ to 7 terms

$= \mathbf{2187}$

(c) $2 \times 2^2 \times 2^5 = 2^{(1+2+5)} = 2^8 = \mathbf{256}$

Problem 2. Find the value of: (a) $\dfrac{7^5}{7^3}$ and (b) $\dfrac{5^7}{5^4}$

From law (ii):

(a) $\dfrac{7^5}{7^3} = 7^{(5-3)} = 7^2 = \mathbf{49}$

(b) $\dfrac{5^7}{5^4} = 5^{(7-4)} = 5^3 = \mathbf{125}$

Problem 3. Evaluate: (a) $5^2 \times 5^3 \div 5^4$ and (b) $(3 \times 3^5) \div (3^2 \times 3^3)$

From laws (i) and (ii):

(a) $5^2 \times 5^3 \div 5^4 = \dfrac{5^2 \times 5^3}{5^4} = \dfrac{5^{(2+3)}}{5^4}$

$= \dfrac{5^5}{5^4} = 5^{(5-4)} = 5^1 = \mathbf{5}$

(b) $(3 \times 3^5) \div (3^2 \times 3^3) = \dfrac{3 \times 3^5}{3^2 \times 3^3} = \dfrac{3^{(1+5)}}{3^{(2+3)}}$

$= \dfrac{3^6}{3^5} = 3^{6-5} = 3^1 = \mathbf{3}$

Problem 4. Simplify: (a) $(2^3)^4$ (b) $(3^2)^5$, expressing the answers in index form.

From law (iii):

(a) $(2^3)^4 = 2^{3\times4} = \mathbf{2^{12}}$

(b) $(3^2)^5 = 3^{2\times5} = \mathbf{3^{10}}$

Problem 5. Evaluate: $\dfrac{(10^2)^3}{10^4 \times 10^2}$

From the laws of indices:

$$\frac{(10^2)^3}{10^4 \times 10^2} = \frac{10^{(2\times3)}}{10^{(4+2)}} = \frac{10^6}{10^6} = 10^{6-6} = 10^0 = \mathbf{1}$$

Problem 6. Find the value of (a) $\dfrac{2^3 \times 2^4}{2^7 \times 2^5}$ and (b) $\dfrac{(3^2)^3}{3 \times 3^9}$

From the laws of indices:

(a) $\dfrac{2^3 \times 2^4}{2^7 \times 2^5} = \dfrac{2^{(3+4)}}{2^{(7+5)}} = \dfrac{2^7}{2^{12}} = 2^{7-12} = 2^{-5} = \dfrac{1}{2^5} = \mathbf{\dfrac{1}{32}}$

(b) $\dfrac{(3^2)^3}{3 \times 3^9} = \dfrac{3^{2\times3}}{3^{1+9}} = \dfrac{3^6}{3^{10}} = 3^{6-10} = 3^{-4} = \dfrac{1}{3^4} = \mathbf{\dfrac{1}{81}}$

Problem 7. Evaluate (a) $4^{1/2}$ (b) $16^{3/4}$ (c) $27^{2/3}$ (d) $9^{-1/2}$

(a) $4^{1/2} = \sqrt{4} = \mathbf{\pm 2}$

(b) $16^{3/4} = \sqrt[4]{16^3} = (2)^3 = \mathbf{8}$

(Note that it does not matter whether the 4th root of 16 is found first or whether 16 cubed is found first–the same answer will result.)

(c) $27^{2/3} = \sqrt[3]{27^2} = (3)^2 = \mathbf{9}$

(d) $9^{-1/2} = \dfrac{1}{9^{1/2}} = \dfrac{1}{\sqrt{9}} = \dfrac{1}{\pm 3} = \mathbf{\pm\dfrac{1}{3}}$

Now try the following exercise

In Problems 1 to 12, simplify the expressions given, expressing the answers in index form and with positive indices:

1. (a) $3^3 \times 3^4$ (b) $4^2 \times 4^3 \times 4^4$
2. (a) $2^3 \times 2 \times 2^2$ (b) $7^2 \times 7^4 \times 7 \times 7^3$
3. (a) $\dfrac{2^4}{2^3}$ (b) $\dfrac{3^7}{3^2}$
4. (a) $5^6 \div 5^3$ (b) $7^{13}/7^{10}$
5. (a) $(7^2)^3$ (b) $(3^3)^2$
6. (a) $(15^3)^5$ (b) $(17^2)^4$
7. (a) $\dfrac{2^2 \times 2^3}{2^4}$ (b) $\dfrac{3^7 \times 3^4}{3^5}$
8. (a) $\dfrac{5^7}{5^2 \times 5^3}$ (b) $\dfrac{13^5}{13 \times 13^2}$
9. (a) $\dfrac{(9 \times 3^2)^3}{(3 \times 27)^2}$ (b) $\dfrac{(16 \times 4)^2}{(2 \times 8)^3}$
10. (a) $\dfrac{5^{-2}}{5^{-4}}$ (b) $\dfrac{3^2 \times 3^{-4}}{3^3}$

11. (a) $\dfrac{7^2 \times 7^{-3}}{7 \times 7^{-4}}$ (b) $\dfrac{2^3 \times 2^{-4} \times 2^5}{2 \times 2^{-2} \times 2^6}$

12. (a) $13 \times 13^{-2} \times 13^4 \times 13^{-3}$ (b) $\dfrac{5^{-7} \times 5^2}{5^{-8} \times 5^3}$

3.3 Further worked problems on indices

Problem 8. Evaluate $\dfrac{3^3 \times 5^7}{5^3 \times 3^4}$

The laws of indices only apply to terms **having the same base**. Grouping terms having the same base, and then applying the laws of indices to each of the groups independently gives:

$$\frac{3^3 \times 5^7}{5^3 \times 3^4} = \frac{3^3}{3^4} \times \frac{5^7}{5^3} = 3^{(3-4)} \times 5^{(7-3)}$$

$$= 3^{-1} \times 5^4 = \frac{5^4}{3^1} = \frac{625}{3} = \mathbf{208\frac{1}{3}}$$

Problem 9. Find the value of $\dfrac{2^3 \times 3^5 \times (7^2)^2}{7^4 \times 2^4 \times 3^3}$

$$\frac{2^3 \times 3^5 \times (7^2)^2}{7^4 \times 2^4 \times 3^3} = 2^{3-4} \times 3^{5-3} \times 7^{2\times 2-4}$$

$$= 2^{-1} \times 3^2 \times 7^0 = \frac{1}{2} \times 3^2 \times 1$$

$$= \frac{9}{2} = \mathbf{4\frac{1}{2}}$$

Problem 10. Evaluate: $\dfrac{4^{1.5} \times 8^{1/3}}{2^2 \times 32^{-2/5}}$

$4^{1.5} = 4^{3/2} = \sqrt{4^3} = 2^3 = 8,\ 8^{1/3} = \sqrt[3]{8} = 2,\ 2^2 = 4$

$$32^{-2/5} = \frac{1}{32^{2/5}} = \frac{1}{\sqrt[5]{32^2}} = \frac{1}{2^2} = \frac{1}{4}$$

Hence $\dfrac{4^{1.5} \times 8^{1/3}}{2^2 \times 32^{-2/5}} = \dfrac{8 \times 2}{4 \times \frac{1}{4}} = \dfrac{16}{1} = \mathbf{16}$

Alternatively,

$$\frac{4^{1.5} \times 8^{1/3}}{2^2 \times 32^{-2/5}} = \frac{[(2)^2]^{3/2} \times (2^3)^{1/3}}{2^2 \times (2^5)^{-2/5}} = \frac{2^3 \times 2^1}{2^2 \times 2^{-2}}$$

$$= 2^{3+1-2-(-2)} = 2^4 = \mathbf{16}$$

Problem 11. Evaluate: $\dfrac{3^2 \times 5^5 + 3^3 \times 5^3}{3^4 \times 5^4}$

Dividing each term by the HCF (i.e. highest common factor) of the three terms, i.e. $3^2 \times 5^3$, gives:

$$\frac{3^2 \times 5^5 + 3^3 \times 5^3}{3^4 \times 5^4} = \frac{\dfrac{3^2 \times 5^5}{3^2 \times 5^3} + \dfrac{3^3 \times 5^3}{3^2 \times 5^3}}{\dfrac{3^4 \times 5^4}{3^2 \times 5^3}}$$

$$= \frac{3^{(2-2)} \times 5^{(5-3)} + 3^{(3-2)} \times 5^0}{3^{(4-2)} \times 5^{(4-3)}}$$

$$= \frac{3^0 \times 5^2 + 3^1 \times 5^0}{3^2 \times 5^1}$$

$$= \frac{1 \times 25 + 3 \times 1}{9 \times 5} = \mathbf{\frac{28}{45}}$$

Problem 12. Find the value of $\dfrac{3^2 \times 5^5}{3^4 \times 5^4 + 3^3 \times 5^3}$

To simplify the arithmetic, each term is divided by the HCF of all the terms, i.e. $3^2 \times 5^3$. Thus

$$\frac{3^2 \times 5^5}{3^4 \times 5^4 + 3^3 \times 5^3} = \frac{\dfrac{3^2 \times 5^5}{3^2 \times 5^3}}{\dfrac{3^4 \times 5^4}{3^2 \times 5^3} + \dfrac{3^3 \times 5^3}{3^2 \times 5^3}}$$

$$= \frac{3^{(2-2)} \times 5^{(5-3)}}{3^{(4-2)} \times 5^{(4-3)} + 3^{(3-2)} \times 5^{(3-3)}}$$

$$= \frac{3^0 \times 5^2}{3^2 \times 5^1 + 3^1 \times 5^0}$$

$$= \frac{25}{45 + 3} = \mathbf{\frac{25}{48}}$$

Problem 13. Simplify $\dfrac{7^{-3} \times 3^4}{3^{-2} \times 7^5 \times 5^{-2}}$, expressing the answer in index form with positive indices.

Since $7^{-3} = \dfrac{1}{7^3}$, $\dfrac{1}{3^{-2}} = 3^2$ and $\dfrac{1}{5^{-2}} = 5^2$ then

$$\frac{7^{-3} \times 3^4}{3^{-2} \times 7^5 \times 5^{-2}} = \frac{3^4 \times 3^2 \times 5^2}{7^3 \times 7^5}$$

$$= \frac{3^{(4+2)} \times 5^2}{7^{(3+5)}} = \mathbf{\frac{3^6 \times 5^2}{7^8}}$$

Problem 14. Simplify $\dfrac{16^2 \times 9^{-2}}{4 \times 3^3 - 2^{-3} \times 8^2}$ expressing the answer in index form with positive indices.

Expressing the numbers in terms of their lowest prime numbers gives:

$$\frac{16^2 \times 9^{-2}}{4 \times 3^3 - 2^{-3} \times 8^2} = \frac{(2^4)^2 \times (3^2)^{-2}}{2^2 \times 3^3 - 2^{-3} \times (2^3)^2}$$

$$= \frac{2^8 \times 3^{-4}}{2^2 \times 3^3 - 2^{-3} \times 2^6}$$

$$= \frac{2^8 \times 3^{-4}}{2^2 \times 3^3 - 2^3}$$

Dividing each term by the HCF (i.e. 2^2) gives:

$$\frac{2^8 \times 3^{-4}}{2^2 \times 3^3 - 2^3} = \frac{2^6 \times 3^{-4}}{3^3 - 2} = \mathbf{\frac{2^6}{3^4(3^3 - 2)}}$$

Problem 15. Simplify

$$\frac{\left(\dfrac{4}{3}\right)^3 \times \left(\dfrac{3}{5}\right)^{-2}}{\left(\dfrac{2}{5}\right)^{-3}}$$

giving the answer with positive indices.

A fraction raised to a power means that both the numerator and the denominator of the fraction are raised to that power, i.e. $\left(\dfrac{4}{3}\right)^3 = \dfrac{4^3}{3^3}$

A fraction raised to a negative power has the same value as the inverse of the fraction raised to a positive power.

Thus, $$\left(\frac{3}{5}\right)^{-2} = \frac{1}{\left(\dfrac{3}{5}\right)^2} = \frac{1}{\dfrac{3^2}{5^2}} = 1 \times \frac{5^2}{3^2} = \frac{5^2}{3^2}$$

Similarly, $$\left(\frac{2}{5}\right)^{-3} = \left(\frac{5}{2}\right)^3 = \frac{5^3}{2^3}$$

Thus, $$\frac{\left(\dfrac{4}{3}\right)^3 \times \left(\dfrac{3}{5}\right)^{-2}}{\left(\dfrac{2}{5}\right)^{-3}} = \frac{\dfrac{4^3}{3^3} \times \dfrac{5^2}{3^2}}{\dfrac{5^3}{2^3}}$$

$$= \frac{4^3}{3^3} \times \frac{5^2}{3^2} \times \frac{2^3}{5^3}$$

$$= \frac{(2^2)^3 \times 2^3}{3^{(3+2)} \times 5^{(3-2)}} = \mathbf{\frac{2^9}{3^5 \times 5}}$$

Now try the following exercise

In Problems 1 and 2, simplify the expressions given, expressing the answers in index form and with positive indices:

1. (a) $\dfrac{3^3 \times 5^2}{5^4 \times 3^4}$ (b) $\dfrac{7^{-2} \times 3^{-2}}{3^5 \times 7^4 \times 7^{-3}}$
2. (a) $\dfrac{4^2 \times 9^3}{8^3 \times 3^4}$ (b) $\dfrac{8^{-2} \times 5^2 \times 3^{-4}}{25^2 \times 2^4 \times 9^{-2}}$
3. Evaluate (a) $\left(\dfrac{1}{3^2}\right)^{-1}$ (b) $81^{0.25}$ (c) $16^{(-1/4)}$ (d) $\left(\dfrac{4}{9}\right)^{1/2}$

In problems 4 to 10, evaluate the expressions given.

4. $\dfrac{9^2 \times 7^4}{3^4 \times 7^4 + 3^3 \times 7^2}$
5. $\dfrac{3^3 \times 5^2}{2^3 \times 3^2 - 8^2 \times 9}$
6. $\dfrac{3^3 \times 7^2 - 5^2 \times 7^3}{3^2 \times 5 \times 7^2}$
7. $\dfrac{(2^4)^2 - 3^{-2} \times 4^4}{2^3 \times 16^2}$
8. $\dfrac{\left(\frac{1}{2}\right)^3 - \left(\frac{2}{3}\right)^{-2}}{\left(\frac{3}{5}\right)^2}$
9. $\dfrac{\left(\frac{4}{3}\right)^4}{\left(\frac{2}{9}\right)^2}$
10. $\dfrac{(3^2)^{3/2} \times (8^{1/3})^2}{(3)^2 \times (4^3)^{1/2} \times (9)^{-1/2}}$

6.2 Laws of Indices

The laws of indices are:

(i) $a^m \times a^n = a^{m+n}$

(ii) $\dfrac{a^m}{a^n} = a^{m-n}$

(iii) $(a^m)^n = a^{mn}$

(iv) $a^{m/n} = \sqrt[n]{a^m}$

(v) $a^{-n} = \dfrac{1}{a^n}$

(vi) $a^0 = 1$

Quadratic equations

10.1 Introduction to quadratic equations

An **equation** is a statement that two quan-tities are equal and to **'solve an equation'** means 'to find the value of the unknown'. The value of the unknown is called the **root** of the equation.

A **quadratic equation** is one in which the highest power of the unknown quantity is 2. For example, $x^2 - 3x + 1 = 0$ is a quadratic equation.

There are four methods of **solving quadratic equations**.

These are: (i) by factorization (where possible)

(ii) by 'completing the square'

(iii) by using the 'quadratic formula'

or (iv) graphically

10.2 Solution of quadratic equations by factorization

Multiplying out $(2x+1)(x-3)$ gives $2x^2 - 6x + x - 3$, i.e. $2x^2 - 5x - 3$. The reverse process of moving from $2x^2 - 5x - 3$ to $(2x+1)(x-3)$ is called **factorizing**.

If the quadratic expression can be factorized this provides the simplest method of solving a quadratic equation.

For example, if $2x^2 - 5x - 3 = 0$, then, by factorizing:

$$(2x+1)(x-3) = 0$$

Hence either $(2x+1) = 0$ i.e. $x = -\frac{1}{2}$

or $(x-3) = 0$ i.e. $x = 3$

The technique of factorizing is often one of 'trial and error'.

Problem 1. Solve the equations (a) $x^2 + 2x - 8 = 0$ (b) $3x^2 - 11x - 4 = 0$ by factorization.

(a) $x^2 + 2x - 8 = 0$. The factors of x^2 are x and x. These are placed in brackets thus: $(x \quad)(x \quad)$

The factors of -8 are $+8$ and -1, or -8 and $+1$, or $+4$ and -2, or -4 and $+2$. The only combination to give a middle term of $+2x$ is $+4$ and -2, i.e.

$$x^2 + 2x - 8 = (x+4)(x-2)$$

(Note that the product of the two inner terms added to the product of the two outer terms must equal the middle term, $+2x$ in this case.)

The quadratic equation $x^2 + 2x - 8 = 0$ thus becomes $(x+4)(x-2) = 0$.

Since the only way that this can be true is for either the first or the second, or both factors to be zero, then

either $(x+4) = 0$ i.e. $x = -4$

or $(x-2) = 0$ i.e. $x = 2$

Hence the roots of $x^2 + 2x - 8 = 0$ are $x = -4$ and 2

(b) $3x^2 - 11x - 4 = 0$

The factors of $3x^2$ are $3x$ and x. These are placed in brackets thus: $(3x \quad)(x \quad)$

The factors of -4 are -4 and $+1$, or $+4$ and -1, or -2 and 2.

Remembering that the product of the two inner terms added to the product of the two outer terms must equal $-11x$, the only combination to give this is $+1$ and -4, i.e.

$$3x^2 - 11x - 4 = (3x+1)(x-4)$$

The quadratic equation $3x^2 - 11x - 4 = 0$ thus becomes $(3x+1)(x-4) = 0$

Hence, either $(3x+1) = 0$ i.e. $\boldsymbol{x = -\frac{1}{3}}$

or $(x-4) = 0$ i.e. $\boldsymbol{x = 4}$

and both solutions may be checked in the original equation.

Problem 2. Determine the roots of (a) $x^2 - 6x + 9 = 0$, and (b) $4x^2 - 25 = 0$, by factorization.

(a) $x^2 - 6x + 9 = 0$. Hence $(x-3)(x-3) = 0$, i.e. $(x-3)^2 = 0$ (the left-hand side is known as **a perfect square**). Hence $\boldsymbol{x = 3}$ is the only root of the equation $x^2 - 6x + 9 = 0$.

(b) $4x^2 - 25 = 0$ (the left-hand side is **the difference of two squares**, $(2x)^2$ and $(5)^2$). Thus $(2x+5)(2x-5) = 0$

Hence either $(2x+5) = 0$ i.e. $\boldsymbol{x = -\frac{5}{2}}$

or $(2x-5) = 0$ i.e. $\boldsymbol{x = \frac{5}{2}}$

Problem 3. Solve the following quadratic equations by factorizing: (a) $4x^2 + 8x + 3 = 0$ (b) $15x^2 + 2x - 8 = 0$.

(a) $4x^2 + 8x + 3 = 0$. The factors of $4x^2$ are $4x$ and x or $2x$ and $2x$. The factors of 3 are 3 and 1, or -3 and -1. Remembering that the product of the inner terms added to the product of the two outer terms must equal $+8x$, the only combination that is true (by trial and error) is

$$(4x^2 + 8x + 3) = (2x+3)(2x+1)$$

Hence $(2x+3)(2x+1) = 0$ from which, either

$$(2x+3) = 0 \text{ or } (2x+1) = 0$$

Thus $2x = -3$, from which $\boldsymbol{x = -\frac{3}{2}}$

or $2x = -1$, from which $\boldsymbol{x = -\frac{1}{2}}$

which may be checked in the original equation.

(b) $15x^2 + 2x - 8 = 0$. The factors of $15x^2$ are $15x$ and x or $5x$ and $3x$. The factors of -8 are -4 and $+2$, or 4 and -2, or -8 and $+1$, or 8 and -1. By trial and error the only combination that works is

$$15x^2 + 2x - 8 = (5x+4)(3x-2)$$

Hence $(5x+4)(3x-2) = 0$ from which

either $5x + 4 = 0$

or $3x - 2 = 0$

Hence $\boldsymbol{x = -\frac{4}{5}}$ or $\boldsymbol{x = \frac{2}{3}}$

which may be checked in the original equation.

Problem 4. The roots of a quadratic equation are $\frac{1}{3}$ and -2. Determine the equation.

If the roots of a quadratic equation are α and β then $(x-\alpha)(x-\beta) = 0$

Hence if $\alpha = \dfrac{1}{3}$ and $\beta = -2$, then

$$\left(x - \frac{1}{3}\right)(x - (-2)) = 0$$

$$\left(x - \frac{1}{3}\right)(x + 2) = 0$$

$$x^2 - \frac{1}{3}x + 2x - \frac{2}{3} = 0$$

$$x^2 + \frac{5}{3}x - \frac{2}{3} = 0$$

Hence $\boldsymbol{3x^2 + 5x - 2 = 0}$

Problem 5. Find the equations in x whose roots are (a) 5 and -5 (b) 1.2 and -0.4.

(a) If 5 and -5 are the roots of a quadratic equation then

$(x-5)(x+5) = 0$

i.e. $x^2 - 5x + 5x - 25 = 0$

i.e. $\boldsymbol{x^2 - 25} = 0$

(b) If 1.2 and -0.4 are the roots of a quadratic equation then

$(x-1.2)(x+0.4) = 0$

i.e. $x^2 - 1.2x + 0.4x - 0.48 = 0$

i.e. $\boldsymbol{x^2 - 0.8x - 0.48 = 0}$

Now try the following exercise

In Problems 1 to 12, solve the given equations by factorization.

1. $x^2 + 4x - 32 = 0$
2. $x^2 - 16 = 0$
3. $(x+2)^2 = 16$
4. $2x^2 - x - 3 = 0$
5. $6x^2 - 5x + 1 = 0$
6. $10x^2 + 3x - 4 = 0$
7. $x^2 - 4x + 4 = 0$
8. $21x^2 - 25x = 4$
9. $8x^2 + 13x - 6 = 0$
10. $5x^2 + 13x - 6 = 0$
11. $6x^2 - 5x - 4 = 0$
12. $8x^2 + 2x - 15 = 0$

In Problems 13 to 18, determine the quadratic equations in x whose roots are

13. 3 and 1 14. 2 and -5 15. -1 and -4

16. $2\frac{1}{2}$ and $-\frac{1}{2}$ 17. 6 and -6 18. 2.4 and -0.7

10.3 Solution of quadratic equations by 'completing the square'

An expression such as x^2 or $(x+2)^2$ or $(x-3)^2$ is called a perfect square.

If $x^2 = 3$ then $x = \pm\sqrt{3}$

If $(x+2)^2 = 5$ then $x+2 = \pm\sqrt{5}$ and $x = -2 \pm \sqrt{5}$

If $(x-3)^2 = 8$ then $x-3 = \pm\sqrt{8}$ and $x = 3 \pm \sqrt{8}$

Hence if a quadratic equation can be rearranged so that one side of the equation is a perfect square and the other side of the equation is a number, then the solution of the equation is readily obtained by taking the square roots of each side as in the above examples. The process of rearranging one side of a quadratic equation into a perfect square before solving is called **'completing the square'**.

$$(x+a)^2 = x^2 + 2ax + a^2$$

Thus in order to make the quadratic expression $x^2 + 2ax$ into a perfect square it is necessary to add (half the coefficient of x)^2 i.e. $\left(\frac{2a}{2}\right)^2$ or a^2

For example, $x^2 + 3x$ becomes a perfect square by adding $\left(\frac{3}{2}\right)^2$, i.e.

$$x^2 + 3x + \left(\frac{3}{2}\right)^2 = \left(x + \frac{3}{2}\right)^2$$

The method is demonstrated in the following worked problems.

Problem 6. Solve $2x^2 + 5x = 3$ by 'completing the square'.

The procedure is as follows:

1. Rearrange the equation so that all terms are on the same side of the equals sign (and the coefficient of the x^2 term is positive).

   Hence $2x^2 + 5x - 3 = 0$

2. Make the coefficient of the x^2 term unity. In this case this is achieved by dividing throughout by 2. Hence

   $$\frac{2x^2}{2} + \frac{5x}{2} - \frac{3}{2} = 0$$

   i.e. $$x^2 + \frac{5}{2}x - \frac{3}{2} = 0$$

3. Rearrange the equations so that the x^2 and x terms are on one side of the equals sign and the constant is on the other side. Hence

   $$x^2 + \frac{5}{2}x = \frac{3}{2}$$

4. Add to both sides of the equation (half the coefficient of x)^2. In this case the coefficient of x is $\frac{5}{2}$. Half the coefficient squared is therefore $\left(\frac{5}{4}\right)^2$. Thus

   $$x^2 + \frac{5}{2}x + \left(\frac{5}{4}\right)^2 = \frac{3}{2} + \left(\frac{5}{4}\right)^2$$

   The LHS is now a perfect square, i.e.

   $$\left(x + \frac{5}{4}\right)^2 = \frac{3}{2} + \left(\frac{5}{4}\right)^2$$

5. Evaluate the RHS. Thus

   $$\left(x + \frac{5}{4}\right)^2 = \frac{3}{2} + \frac{25}{16} = \frac{24+25}{16} = \frac{49}{16}$$

6. Taking the square root of both sides of the equation (remembering that the square root of a number gives a $\pm$ answer). Thus

   $$\sqrt{\left(x + \frac{5}{4}\right)^2} = \sqrt{\left(\frac{49}{16}\right)}$$

   i.e. $$x + \frac{5}{4} = \pm\frac{7}{4}$$

7. Solve the simple equation. Thus

   $$x = -\frac{5}{4} \pm \frac{7}{4}$$

   i.e. $$x = -\frac{5}{4} + \frac{7}{4} = \frac{2}{4} = \frac{1}{2}$$

   and $$x = -\frac{5}{4} - \frac{7}{4} = -\frac{12}{4} = -3$$

   Hence $\boldsymbol{x = \frac{1}{2}}$ or $\mathbf{-3}$ are the roots of the equation $2x^2 + 5x = 3$

Problem 7. Solve $2x^2 + 9x + 8 = 0$, correct to 3 significant figures, by 'completing the square'.

Making the coefficient of x^2 unity gives:

$$x^2 + \frac{9}{2}x + 4 = 0$$

and rearranging gives: $x^2 + \frac{9}{2}x = -4$

Adding to both sides (half the coefficient of $x)^2$ gives:

$$x^2 + \frac{9}{2}x + \left(\frac{9}{4}\right)^2 = \left(\frac{9}{4}\right)^2 - 4$$

The LHS is now a perfect square, thus

$$\left(x + \frac{9}{4}\right)^2 = \frac{81}{16} - 4 = \frac{17}{16}$$

Taking the square root of both sides gives:

$$x + \frac{9}{4} = \sqrt{\left(\frac{17}{16}\right)} = \pm 1.031$$

Hence $\quad x = -\frac{9}{4} \pm 1.031$

i.e. $\boldsymbol{x = -1.22}$ or $\boldsymbol{-3.28}$, correct to 3 significant figures.

Problem 8. By 'completing the square', solve the quadratic equation $4.6y^2 + 3.5y - 1.75 = 0$, correct to 3 decimal places.

$$4.6y^2 + 3.5y - 1.75 = 0$$

Making the coefficient of y^2 unity gives:

$$y^2 + \frac{3.5}{4.6}y - \frac{1.75}{4.6} = 0$$

and rearranging gives:

$$y^2 + \frac{3.5}{4.6}y = \frac{1.75}{4.6}$$

Adding to both sides (half the coefficient of $y)^2$ gives:

$$y^2 + \frac{3.5}{4.6}y + \left(\frac{3.5}{9.2}\right)^2 = \frac{1.75}{4.6} + \left(\frac{3.5}{9.2}\right)^2$$

The LHS is now a perfect square, thus

$$\left(y + \frac{3.5}{9.2}\right)^2 = 0.5251654$$

Taking the square root of both sides gives:

$$y + \frac{3.5}{9.2} = \sqrt{0.5251654} = \pm 0.7246830$$

Hence $\quad y = -\frac{3.5}{9.2} \pm 0.7246830$

i.e. $\quad \boldsymbol{y = 0.344}$ or $\boldsymbol{-1.105}$

Now try the following exercise

In Problems 1 to 6, solve the given equations by completing the square, each correct to 3 decimal places.

1. $x^2 + 4x + 1 = 0$
2. $2x^2 + 5x - 4 = 0$
3. $3x^2 - x - 5 = 0$
4. $5x^2 - 8x + 2 = 0$
5. $4x^2 - 11x + 3 = 0$
6. $2x^2 + 5x = 2$

10.4 Solution of quadratic equations by formula

Let the general form of a quadratic equation be given by:

$$ax^2 + bx + c = 0 \quad \text{where } a, b \text{ and } c \text{ are constants.}$$

Dividing $ax^2 + bx + c = 0$ by a gives:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Rearranging gives:

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Adding to each side of the equation the square of half the coefficient of the term in x to make the LHS a perfect square gives:

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

Rearranging gives:

$$\left(x + \frac{b}{a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}$$

Taking the square root of both sides gives:

$$x + \frac{b}{2a} = \sqrt{\left(\frac{b^2 - 4ac}{4a^2}\right)} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

Hence $\quad x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$

i.e. the quadratic formula is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(This method of solution is 'completing the square' – as shown in Section 10.3.)

Summarizing:

if $ax^2 + bx + c = 0$ then $\boxed{\boldsymbol{x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}}}$

This is known as the **quadratic formula**.

Problem 9. Solve (a) $x^2 + 2x - 8 = 0$ and (b) $3x^2 - 11x - 4 = 0$ by using the quadratic formula.

(a) Comparing $x^2 + 2x - 8 = 0$ with $ax^2 + bx + c = 0$ gives $a = 1$, $b = 2$ and $c = -8$
Substituting these values into the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ gives:}$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-8)}}{2(1)} = \frac{-2 \pm \sqrt{4 + 32}}{2}$$

$$= \frac{-2 \pm \sqrt{36}}{2} = \frac{-2 \pm 6}{2}$$

$$= \frac{-2 + 6}{2} \text{ or } \frac{-2 - 6}{2}$$

Hence $\boldsymbol{x = \frac{4}{2} = 2}$ **or** $\boldsymbol{\frac{-8}{2} = -4}$ (as in Problem 1(a)).

(b) Comparing $3x^2 - 11x - 4 = 0$ with $ax^2 + bx + c = 0$ gives $a = 3$, $b = -11$ and $c = -4$. Hence

$$x = \frac{-(-11) \pm \sqrt{(-11)^2 - 4(3)(-4)}}{2(3)}$$

$$= \frac{+11 \pm \sqrt{121 + 48}}{6} = \frac{11 \pm \sqrt{169}}{6}$$

$$= \frac{11 \pm 13}{6} = \frac{11 + 13}{6} \text{ or } \frac{11 - 13}{6}$$

Hence $\boldsymbol{x = \frac{24}{6} = 4}$ **or** $\boldsymbol{\frac{-2}{6} = -\frac{1}{3}}$ (as in Problem 1(b)).

Problem 10. Solve $4x^2 + 7x + 2 = 0$ giving the roots correct to 2 decimal places.

Comparing $4x^2 + 7x + 2 = 0$ with $ax^2 + bx + c$ gives $a = 4$, $b = 7$ and $c = 2$.

Hence

$$x = \frac{-7 \pm \sqrt{[(7)^2 - 4(4)(2)]}}{2(4)} = \frac{-7 \pm \sqrt{17}}{8}$$

$$= \frac{-7 \pm 4.123}{8} = \frac{-7 + 4.123}{8} \text{ or } \frac{-7 - 4.123}{8}$$

Hence $\boldsymbol{x = -0.36}$ or $\boldsymbol{-1.39}$, correct to 2 decimal places.

Problem 11. Use the quadratic formula to solve $\dfrac{x+2}{4} + \dfrac{3}{x-1} = 7$ correct to 4 significant figures.

Multiplying throughout by $4(x-1)$ gives:

$$4(x-1)\frac{(x+2)}{4} + 4(x-1)\frac{3}{(x-1)} = 4(x-1)(7)$$

i.e. $\quad (x-1)(x+2) + (4)(3) = 28(x-1)$

$$x^2 + x - 2 + 12 = 28x - 28$$

Hence $\quad x^2 - 27x + 38 = 0$

Using the quadratic formula:

$$x = \frac{-(-27) \pm \sqrt{(-27)^2 - 4(1)(38)}}{2}$$

$$= \frac{27 \pm \sqrt{577}}{2} = \frac{27 \pm 24.0208}{2}$$

Hence $\quad x = \dfrac{27 + 24.0208}{2} = 25.5104$

or $\quad x = \dfrac{27 - 24.0208}{2} = 1.4896$

Hence $\boldsymbol{x = 25.51}$ or **1.490**, correct to 4 significant figures.

Now try the following exercise

In Problems 1 to 6 solve the given equations by using the quadratic formula, correct to 3 decimal places.

1. $2x^2 + 5x - 4 = 0$
2. $5.76x^2 + 2.86x - 1.35 = 0$
3. $2x^2 - 7x + 4 = 0$
4. $4x + 5 = \dfrac{3}{x}$
5. $(2x + 1) = \dfrac{5}{x - 3}$
6. $\dfrac{x + 1}{x - 1} = x - 3$

Logarithms

14.1 Introduction to logarithms

With the use of calculators firmly established, logarithmic tables are now rarely used for calculation. However, the theory of logarithms is important, for there are several scientific and engineering laws that involve the rules of logarithms.

If a number y can be written in the form a^x, then the index x is called the 'logarithm of y to the base of a',

i.e. $\boxed{\textbf{if } y = a^x \quad \textbf{then } x = \log_a y}$

Thus, since $1000 = 10^3$, then $3 = \log_{10} 1000$

Check this using the 'log' button on your calculator.

(a) Logarithms having a base of 10 are called **common logarithms** and $\log_{10}$ is usually abbreviated to lg. The following values may be checked by using a calculator:

$$\lg 17.9 = 1.2528\ldots, \lg 462.7 = 2.6652\ldots$$

and $\lg 0.0173 = -1.7619\ldots$

(b) Logarithms having a base of e (where 'e' is a mathematical constant approximately equal to 2.7183) are called **hyperbolic, Napierian** or **natural logarithms**, and $\log_e$ is usually abbreviated to ln. The following values may be checked by using a calculator:

$$\ln 3.15 = 1.1474\ldots, \ln 362.7 = 5.8935\ldots$$

and $\ln 0.156 = -1.8578\ldots$

For more on Napierian logarithms see Chapter 15.

14.2 Laws of logarithms

There are three laws of logarithms, which apply to any base:

(i) To multiply two numbers:

$$\boxed{\log(A \times B) = \log A + \log B}$$

The following may be checked by using a calculator:

$\lg 10 = 1$,

also $\lg 5 + \lg 2 = 0.69897\ldots + 0.301029\ldots = 1$

Hence $\lg(5 \times 2) = \lg 10 = \lg 5 + \lg 2$

(ii) To divide two numbers:

$$\boxed{\log\left(\frac{A}{B}\right) = \log A - \log B}$$

The following may be checked using a calculator:

$$\ln\left(\frac{5}{2}\right) = \ln 2.5 = 0.91629\ldots$$

Also $\ln 5 - \ln 2 = 1.60943\ldots - 0.69314\ldots$
$= 0.91629\ldots$

Hence $\ln\left(\frac{5}{2}\right) = \ln 5 - \ln 2$

(iii) To raise a number to a power:

$$\boxed{\log A^n = n \log A}$$

The following may be checked using a calculator:

$\lg 5^2 = \lg 25 = 1.39794\ldots$

Also $2 \lg 5 = 2 \times 0.69897\ldots = 1.39794\ldots$

Hence $\lg 5^2 = 2 \lg 5$

Problem 1. Evaluate (a) $\log_3 9$ (b) $\log_{10} 10$ (c) $\log_{16} 8$

(a) Let $x = \log_3 9$ then $3^x = 9$ from the definition of a logarithm, i.e. $3^x = 3^2$, from which $x = 2$.

Hence $\mathbf{\log_3 9 = 2}$

(b) Let $x = \log_{10} 10$ then $10^x = 10$ from the definition of a logarithm, i.e. $10^x = 10^1$, from which $x = 1$.

Hence $\mathbf{\log_{10} 10 = 1}$ (which may be checked by a calculator).

(c) Let $x = \log_{16} 8$ then $16^x = 8$, from the definition of a logarithm, i.e. $(2^4)^x = 2^3$, i.e. $2^{4x} = 2^3$ from the laws of indices, from which, $4x = 3$ and $x = \frac{3}{4}$

Hence $\mathbf{\log_{16} 8 = \frac{3}{4}}$

Problem 2. Evaluate (a) $\lg 0.001$ (b) $\ln e$ (c) $\log_3 \dfrac{1}{81}$

(a) Let $x = \lg 0.001 = \log_{10} 0.001$ then $10^x = 0.001$, i.e. $10^x = 10^{-3}$, from which $x = -3$

Hence $\mathbf{\lg 0.001 = -3}$ (which may be checked by a calculator).

(b) Let $x = \ln e = \log_e e$ then $e^x = e$, i.e. $e^x = e^1$ from which $x = 1$

Hence $\mathbf{\ln e = 1}$ (which may be checked by a calculator).

(c) Let $x = \log_3 \dfrac{1}{81}$ then $3^x = \dfrac{1}{81} = \dfrac{1}{3^4} = 3^{-4}$, from which $x = -4$

Hence $\mathbf{\log_3 \dfrac{1}{81} = -4}$

Problem 3. Solve the following equations:
(a) $\lg x = 3$ (b) $\log_2 x = 3$ (c) $\log_5 x = -2$

(a) If $\lg x = 3$ then $\log_{10} x = 3$ and $x = 10^3$, i.e. $\boldsymbol{x = 1000}$

(b) If $\log_2 x = 3$ then $\boldsymbol{x = 2^3 = 8}$

(c) If $\log_5 x = -2$ then $x = 5^{-2} = \dfrac{1}{5^2} = \mathbf{\dfrac{1}{25}}$

Problem 4. Write (a) log 30 (b) log 450 in terms of log 2, log 3 and log 5 to any base.

(a)
$$\begin{aligned}\log 30 &= \log(2 \times 15) = \log(2 \times 3 \times 5)\\ &= \mathbf{\log 2 + \log 3 + \log 5}\\ &\quad \text{by the first law of logarithms}\end{aligned}$$

(b)
$$\begin{aligned}\log 450 &= \log(2 \times 225) = \log(2 \times 3 \times 75)\\ &= \log(2 \times 3 \times 3 \times 25)\\ &= \log(2 \times 3^2 \times 5^2)\\ &= \log 2 + \log 3^2 + \log 5^2\\ &\quad \text{by the first law of logarithms}\end{aligned}$$

i.e
$$\begin{aligned}\log 450 &= \mathbf{\log 2 + 2\log 3 + 2\log 5}\\ &\quad \text{by the third law of logarithms}\end{aligned}$$

Problem 5. Write $\log\left(\dfrac{8 \times \sqrt[4]{5}}{81}\right)$ in terms of log 2, log 3 and log 5 to any base.

$$\begin{aligned}\log\left(\frac{8 \times \sqrt[4]{5}}{81}\right) &= \log 8 + \log \sqrt[4]{5} - \log 81, \text{ by the first}\\ &\quad \text{and second laws of logarithms}\\ &= \log 2^3 + \log 5^{(1/4)} - \log 3^4\\ &\quad \text{by the laws of indices}\end{aligned}$$

i.e
$$\begin{aligned}\log\left(\frac{8 \times \sqrt[4]{5}}{81}\right) &= \mathbf{3\log 2 + \frac{1}{4}\log 5 - 4\log 3}\\ &\quad \text{by the third law of logarithms}\end{aligned}$$

Problem 6. Simplify $\log 64 - \log 128 + \log 32$

$64 = 2^6$, $128 = 2^7$ and $32 = 2^5$

Hence $\log 64 - \log 128 + \log 32 = \log 2^6 - \log 2^7 + \log 2^5$
$$\begin{aligned}&= 6\log 2 - 7\log 2 + 5\log 2\\ &\quad \text{by the third law of logarithms}\\ &= \mathbf{4\log 2}\end{aligned}$$

Problem 7. Evaluate $\dfrac{\log 25 - \log 125 + \frac{1}{2}\log 625}{3\log 5}$

$$\begin{aligned}\frac{\log 25 - \log 125 + \frac{1}{2}\log 625}{3\log 5} &= \frac{\log 5^2 - \log 5^3 + \frac{1}{2}\log 5^4}{3\log 5}\\ &= \frac{2\log 5 - 3\log 5 + \frac{4}{2}\log 5}{3\log 5}\\ &= \frac{1\log 5}{3\log 5} = \mathbf{\frac{1}{3}}\end{aligned}$$

Problem 8. Solve the equation:
$\log(x-1) + \log(x+1) = 2\log(x+2)$

$$\begin{aligned}\log(x-1) + \log(x+1) &= \log(x-1)(x+1) \text{ from the first}\\ &\quad \text{law of logarithms}\\ &= \log(x^2 - 1)\\ 2\log(x+2) &= \log(x+2)^2 = \log(x^2 + 4x + 4)\end{aligned}$$

Hence if $\log(x^2 - 1) = \log(x^2 + 4x + 4)$
then $(x^2 - 1) = x^2 + 4x + 4$
i.e. $-1 = 4x + 4$

i.e. $\quad -5 = 4x$

i.e. $\quad \boldsymbol{x = -\frac{5}{4}}$ or $\boldsymbol{-1\frac{1}{4}}$

Now try the following exercise

In Problems 1 to 11, evaluate the given expression:

1. $\log_{10} 10\,000$ 2. $\log_2 16$ 3. $\log_5 125$

4. $\log_2 \frac{1}{8}$ 5. $\log_8 2$ 6. $\log_7 343$

7. $\lg 100$ 8. $\lg 0.01$ 9. $\log_4 8$

10. $\log_{27} 3$ 11. $\ln e^2$

In Problems 12 to 18 solve the equations:

12. $\log_{10} x = 4$ 13. $\log x = 5$

14. $\log_3 x = 2$ 15. $\log_4 x = -2\frac{1}{2}$

16. $\lg x = -2$ 17. $\log_8 x = -\frac{4}{3}$

18. $\ln x = 3$

In Problems 19 to 22 write the given expressions in terms of log 2, log 3 and log 5 to any base:

19. $\log 60$ 20. $\log 300$

21. $\log\left(\frac{16 \times \sqrt[4]{5}}{27}\right)$ 22. $\log\left(\frac{125 \times \sqrt[4]{16}}{\sqrt[4]{81^3}}\right)$

Simplify the expressions given in Problems 23 to 25:

23. $\log 27 - \log 9 + \log 81$

24. $\log 64 + \log 32 - \log 128$

25. $\log 8 - \log 4 + \log 32$

Evaluate the expressions given in Problems 26 and 27:

26. $\dfrac{\frac{1}{2}\log 16 - \frac{1}{3}\log 8}{\log 4}$

27. $\dfrac{\log 9 - \log 3 + \frac{1}{2}\log 81}{2\log 3}$

Solve the equations given in Problems 28 to 30:

28. $\log x^4 - \log x^3 = \log 5x - \log 2x$

29. $\log 2t^3 - \log t = \log 16 + \log t$

30. $2\log b^2 - 3\log b = \log 8b - \log 4b$

Number sequences

29.1 Simple sequences

A set of numbers which are connected by a definite law is called a **series** or a **sequence of numbers**. Each of the numbers in the series is called a **term** of the series.

For example, 1, 3, 5, 7, ... is a series obtained by adding 2 to the previous term, and 2, 8, 32, 128, ... is a sequence obtained by multiplying the previous term by 4.

Problem 1. Determine the next two terms in the series: 3, 6, 9, 12, ...

We notice that the sequence 3, 6, 9, 12, ... progressively increases by 3, thus the next two terms will be **15 and 18**.

Problem 2. Find the next three terms in the series: 9, 5, 1, ...

We notice that each term in the series 9, 5, 1, ... progressively decreases by 4, thus the next two terms will be $1 - 4$, i.e. $\mathbf{-3}$ and $-3 - 4$, i.e. $\mathbf{-7}$.

Problem 3. Determine the next two terms in the series: 2, 6, 18, 54, ...

We notice that the second term, 6, is three times the first term, the third term, 18, is three times the second term, and that the fourth term, 54, is three times the third term. Hence the fifth term will be $3 \times 54 = \mathbf{162}$ and the sixth term will be $3 \times 162 = \mathbf{486}$

Now try the following exercise

Determine the next two terms in each of the following series:

1. 5, 9, 13, 17, ...
2. 3, 6, 12, 24, ...
3. 112, 56, 28, ...
4. 12, 7, 2, ...
5. 2, 5, 10, 17, 26, 37, ...
6. 1, 0.1, 0.01, ...
7. 4, 9, 19, 34, ...

29.2 The n'th term of a series

If a series is represented by a general expression, say, $2n + 1$, where n is an integer (i.e. a whole number), then by substituting $n = 1, 2, 3, \ldots$ the terms of the series can be determined; in this example, the first three terms will be:

$$2(1) + 1, 2(2) + 1, 2(3) + 1, \ldots, \quad \text{i.e.} \quad 3, 5, 7, \ldots$$

What is the n'th term of the sequence 1, 3, 5, 7, ... ? Firstly, we notice that the gap between each term is 2, hence the law relating the numbers is:

'$2n$ + something'

The second term, $\quad 3 = 2n + \text{something}$,

hence when $n = 2$ (i.e. the second term of the series), then $3 = 4 + \text{something}$ and the 'something' must be -1. Thus **the n'th term of 1, 3, 5, 7, ... is $\mathbf{2n - 1}$**. Hence the fifth term is given by $2(5) - 1 = 9$, and the twentieth term is $2(20) - 1 = 39$, and so on.

Problem 4. The n'th term of a sequence is given by $3n + 1$. Write down the first four terms.

The first four terms of the series $3n+1$ will be:

$$3(1)+1, 3(2)+1, 3(3)+1 \quad \text{and} \quad 3(4)+1$$

i.e. **4, 7, 10 and 13**

Problem 5. The n'th term of a series is given by $4n-1$. Write down the first four terms.

The first four terms on the series $4n-1$ will be:

$$4(1)-1, 4(2)-1, 4(3)-1 \quad \text{and} \quad 4(4)-1$$

i.e. **3, 7, 11 and 15**

Problem 6. Find the n'th term of the series: 1, 4, 7, ...

We notice that the gap between each of the given three terms is 3, hence the law relating the numbers is:

'$3n +$ something'

The second term, $4 = 3n +$ something,

so when $n = 2$, then $4 = 6 +$ something,

so the 'something' must be -2 (from simple equations).

Thus **the n'th term of the series 1, 4, 7, ... is: $3n-2$**

Problem 7. Find the n'th term of the sequence: 3, 9, 15, 21, ... Hence determine the 15th term of the series.

We notice that the gap between each of the given four terms is 6, hence the law relating the numbers is:

'$6n +$ something'

The second term, $9 = 6n +$ something,

so when $n = 2$, then $9 = 12 +$ something,

so the 'something' must be -3

Thus **the n'th term of the series 3, 9, 15, 21, ... is: $6n-3$**

The 15th term of the series is given by $6n-3$ when $n = 15$.

Hence the 15th term of the series 3, 9, 15, 21, ... is:
$6(15) - 3 =$ **87**

Problem 8. Find the n'th term of the series: 1, 4, 9, 16, 25, ...

This is a special series and does not follow the pattern of the previous examples. Each of the terms in the given series are **square numbers**,

i.e. $1, 4, 9, 16, 25, \ldots \equiv 1^2, 2^2, 3^2, 4^2, 5^2, \ldots$

Hence the n'th term is: n^2

Now try the following exercise

1. The n'th term of a sequence is given by $2n-1$. Write down the first four terms.
2. The n'th term of a sequence is given by $3n+4$. Write down the first five terms.
3. Write down the first four terms of the sequence given by $5n+1$

Find the n'th term in the following series:

4. 5, 10, 15, 20, ...
5. 4, 10, 16, 22, ...
6. 3, 5, 7, 9, ...
7. 2, 6, 10, 14, ...
8. 9, 12, 15, 18, ...
9. 1, 8, 27, 64, 125, ...

29.3 Arithmetic progressions

When a sequence has a constant difference between successive terms it is called an **arithmetic progression** (often abbreviated to *AP*).

Examples include:

(i) 1, 4, 7, 10, 13, ... where the **common difference** is 3,

and (ii) $a, a+d, a+2d, a+3d, \ldots$ where the common difference is d.

If the first term of an *AP* is 'a' and the common difference is 'd' then

$$\boxed{\text{the } n'\text{th term is : } \boldsymbol{a+(n-1)d}}$$

In example (i) above, the 7th term is given by $1+(7-1)3 = \mathbf{19}$, which may be readily checked.

The sum S of an *AP* can be obtained by multiplying the average of all the terms by the number of terms.

The average of all the terms $= \dfrac{a+1}{2}$, where 'a' is the first term and l is the last term, i.e. $l = a+(n-1)d$, for n terms.

Hence the sum of n terms,

$$S_n = n\left(\frac{a+1}{2}\right) = \frac{n}{2}\{a+[a+(n-1)d]\}$$

i.e. $$\boxed{\boldsymbol{S_n = \frac{n}{2}[2a+(n-1)d]}}$$

For example, the sum of the first 7 terms of the series 1, 4, 7, 10, 13, ... is given by

$$S_7 = \tfrac{7}{2}[2(1)+(7-1)3], \quad \text{since } a=1 \text{ and } d=3$$
$$= \tfrac{7}{2}[2+18] = \tfrac{7}{2}[20] = \mathbf{70}$$

29.4 Worked problems on arithmetic progression

Problem 9. Determine (a) the ninth, and (b) the sixteenth term of the series 2, 7, 12, 17, ...

2, 7, 12, 17, ... is an arithmetic progression with a common difference, d, of 5

(a) The n'th term of an AP is given by $a+(n-1)d$

Since the first term $a=2, d=5$ and $n=9$

then the 9th term is:

$$2+(9-1)5 = 2+(8)(5) = 2+40 = \mathbf{42}$$

(b) The 16th term is:

$$2+(16-1)5 = 2+(15)(5) = 2+75 = \mathbf{77}$$

Problem 10. The 6th term of an AP is 17 and the 13th term is 38. Determine the 19th term.

The n'th term of an AP is $a+(n-1)d$

The 6th term is: $\quad a+5d=17 \quad (1)$

The 13th term is: $\quad a+12d=38 \quad (2)$

Equation (2) − equation (1) gives: $7d=21$, from which, $d=\frac{21}{7}=3$

Substituting in equation (1) gives: $a+15=17$, from which, $a=2$

Hence the 19th term is:

$$a+(n-1)d = 2+(19-1)3 = 2+(18)(3)$$
$$= 2+54 = \mathbf{56}$$

Problem 11. Determine the number of the term whose value is 22 in the series $2\frac{1}{2}, 4, 5\frac{1}{2}, 7, \ldots$

$2\frac{1}{2}, 4, 5\frac{1}{2}, 7, \ldots$ is an AP where $a=2\frac{1}{2}$ and $d=1\frac{1}{2}$

Hence if the n'th term is 22 then: $a+(n-1)d=22$

i.e. $\quad 2\frac{1}{2}+(n-1)\left(1\frac{1}{2}\right)=22$

$$(n-1)\left(1\frac{1}{2}\right)=22-2\frac{1}{2}=19\frac{1}{2}$$

$$n-1=\frac{19\frac{1}{2}}{1\frac{1}{2}}=13 \quad \text{and } n=13+1=14$$

i.e. **the 14th term of the *AP* is 22**

Problem 12. Find the sum of the first 12 terms of the series 5, 9, 13, 17, ...

5, 9, 13, 17, ... is an AP where $a=5$ and $d=4$

The sum of n terms of an AP,

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

Hence the sum of the first 12 terms,

$$S_{12} = \frac{12}{2}[2(5)+(12-1)4]$$
$$= 6[10+44] = 6(54) = \mathbf{324}$$

Problem 13. Find the sum of the first 21 terms of the series 3.5, 4.1, 4.7, 5.3, ...

3.5, 4.1, 4.7, 5.3, ... is an AP where $a=3.5$ and $d=0.6$

The sum of the first 21 terms,

$$S_{21} = \frac{21}{2}[2a+(n-1)d]$$
$$= \frac{21}{2}[2(3.5)+(21-1)0.6] = \frac{21}{2}[7+12]$$
$$= \frac{21}{2}(19) = \frac{399}{2} = \mathbf{199.5}$$

Now try the following exercise

1. Find the 11th term of the series 8, 14, 20, 26, ...
2. Find the 17th term of the series 11, 10.7, 10.4, 10.1, ...
3. The seventh term of a series is 29 and the eleventh term is 54. Determine the sixteenth term.
4. Find the 15th term of an arithmetic progression of which the first term is $2\frac{1}{2}$ and the tenth term is 16.

5. Determine the number of the term which is 29 in the series 7, 9.2, 11.4, 13.6, ...
6. Find the sum of the first 11 terms of the series 4, 7, 10, 13, ...
7. Determine the sum of the series 6.5, 8.0, 9.5, 11.0, ..., 32

29.5 Further worked problems on arithmetic progressions

Problem 14. The sum of 7 terms of an *AP* is 35 and the common difference is 1.2. Determine the first term of the series.

$n = 7, d = 1.2$ and $S_7 = 35$

Since the sum of n terms of an *AP* is given by

$$S_n = \frac{n}{2}[2a + (n-1)\,\mathrm{d}], \quad \text{then}$$

$$35 = \frac{7}{2}[2a + (7-1)1.2] = \frac{7}{2}[2a + 7.2]$$

Hence $\dfrac{35 \times 2}{7} = 2a + 7.2$

$$10 = 2a + 7.2$$

Thus $2a = 10 - 7.2 = 2.8$, from which $a = \dfrac{2.8}{2} = 1.4$

i.e. **the first term, $\boldsymbol{a = 1.4}$**

Problem 15. Three numbers are in arithmetic progression. Their sum is 15 and their product is 80. Determine the three numbers.

Let the three numbers be $(a - d)$, a and $(a + d)$

Then $(a - d) + a + (a + d) = 15$, i.e. $3a = 15$, from which, $a = 5$

Also, $a(a - d)(a + d) = 80$, i.e. $a(a^2 - d^2) = 80$

Since $a = 5, 5(5^2 - d^2) = 80$

$$125 - 5d^2 = 80$$
$$125 - 80 = 5d^2$$
$$45 = 5d^2$$

from which, $d^2 = \dfrac{45}{5} = 9$. Hence $d = \sqrt{9} = \pm 3$

The three numbers are thus $(5 - 3)$, 5 and $(5 + 3)$, i.e. **2, 5 and 8**

Problem 16. Find the sum of all the numbers between 0 and 207 which are exactly divisible by 3.

The series 3, 6, 9, 12, ... 207 is an *AP* whose first term $a = 3$ and common difference $d = 3$

The last term is $a + (n - 1)d = 207$

i.e. $3 + (n - 1)3 = 207$, from which

$$(n-1) = \frac{207 - 3}{3} = 68$$

Hence $n = 68 + 1 = 69$

The sum of all 69 terms is given by

$$S_{69} = \frac{n}{2}[2a + (n-1)d]$$
$$= \frac{69}{2}[2(3) + (69 - 1)3]$$
$$= \frac{69}{2}[6 + 204] = \frac{69}{2}(210) = \mathbf{7245}$$

Problem 17. The first, twelfth and last term of an arithmetic progression are 4, $31\frac{1}{2}$, and $376\frac{1}{2}$ respectively. Determine (a) the number of terms in the series, (b) the sum of all the terms and (c) the 80'th term.

(a) Let the *AP* be $a, a + d, a + 2d, \ldots, a + (n - 1)d$, where $a = 4$

The 12th term is: $a + (12 - 1)d = 31\frac{1}{2}$

i.e. $4 + 11d = 31\frac{1}{2}$, from which,

$$11d = 31\tfrac{1}{2} - 4 = 27\tfrac{1}{2}$$

Hence $d = \dfrac{27\frac{1}{2}}{11} = 2\dfrac{1}{2}$

The last term is $a + (n - 1)d$

i.e. $4 + (n - 1)\left(2\frac{1}{2}\right) = 376\frac{1}{2}$

$$(n-1) = \frac{376\frac{1}{2} - 4}{2\frac{1}{2}} = \frac{372\frac{1}{2}}{2\frac{1}{2}} = 149$$

Hence the number of terms in the series,

$$\boldsymbol{n = 149 + 1 = 150}$$

(b) Sum of all the terms,

$$S_{150} = \frac{n}{2}[2a + (n-1)d]$$
$$= \frac{150}{2}\left[2(4) + (150 - 1)\left(2\frac{1}{2}\right)\right]$$
$$= 75\left[8 + (149)\left(2\frac{1}{2}\right)\right] = 75[8 + 372.5]$$
$$= 75(380.5) = \mathbf{28537.5}$$

(c) The 80th term is:

$$a + (n-1)d = 4 + (80 - 1)\left(2\tfrac{1}{2}\right)$$
$$= 4 + (79)\left(2\tfrac{1}{2}\right)$$
$$= 4 + 197.5 = \mathbf{201\tfrac{1}{2}}$$

Now try the following exercise

1. The sum of 15 terms of an arithmetic progression is 202.5 and the common difference is 2. Find the first term of the series.
2. Three numbers are in arithmetic progression. Their sum is 9 and their product is $20\frac{1}{4}$. Determine the three numbers.
3. Find the sum of all the numbers between 5 and 250 which are exactly divisible by 4.
4. Find the number of terms of the series 5, 8, 11, ... of which the sum is 1025.
5. Insert four terms between 5 and $22\frac{1}{2}$ to form an arithmetic progression.
6. The first, tenth and last terms of an arithmetic progression are 9, 40.5, and 425.5 respectively. Find (a) the number of terms, (b) the sum of all the terms and (c) the 70th term.
7. On commencing employment a man is paid a salary of £7200 per annum and receives annual increments of £350. Determine his salary in the 9th year and calculate the total he will have received in the first 12 years.
8. An oil company bores a hole 80 m deep. Estimate the cost of boring if the cost is £30 for drilling the first metre with an increase in cost of £2 per metre for each succeeding metre.

29.6 Geometric progressions

When a sequence has a constant ratio between successive terms it is called a **geometric progression** (often abbreviated to *GP*). The constant is called the **common ratio, *r***

Examples include

(i) 1, 2, 4, 8, ... where the common ratio is 2,

and (ii) $a, ar, ar^2, ar^3, \ldots$ where the common ratio is r

If the first term of a *GP* is 'a' and the common ratio is r, then

$$\boxed{\text{the } n'\text{th term is : } \boldsymbol{ar^{n-1}}}$$

which can be readily checked from the above examples.

For example, the 8th term of the *GP* 1, 2, 4, 8, ... is $(1)(2)^7 = \mathbf{128}$, since $a = 1$ and $r = 2$

Let a *GP* be $a, ar, ar^2, ar^3, \ldots ar^{n-1}$

then the sum of n terms,

$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} \cdots \qquad (1)$$

Multiplying throughout by r gives:

$$rS_n = ar + ar^2 + ar^3 + ar^4 + \cdots ar^{n-1} + ar^n \cdots \qquad (2)$$

Subtracting equation (2) from equation (1) gives:

$$S_n - rS_n = a - ar^n$$

i.e. $S_n(1 - r) = a(1 - r^n)$

Thus the sum of n terms, $\boxed{\boldsymbol{S_n = \dfrac{a(1-r^n)}{(1-r)}}}$ which is valid when $r < 1$

Subtracting equation (1) from equation (2) gives

$\boxed{\boldsymbol{S_n = \dfrac{a(r^n-1)}{(r-1)}}}$ which is valid when $r > 1$

For example, the sum of the first 8 terms of the *GP* 1, 2, 4, 8, 16, ... is given by

$$S_8 = \frac{1(2^8 - 1)}{(2 - 1)}, \text{ since } a = 1 \text{ and } r = 2$$

i.e. $S_8 = \dfrac{1(256 - 1)}{1} = \mathbf{255}$

When the common ratio r of a *GP* is less than unity, the sum of n terms,

$$S_n = \frac{a(1 - r^n)}{(1 - r)}, \text{ which may be written as}$$

$$S_n = \frac{a}{(1 - r)} - \frac{ar^n}{(1 - r)}$$

Since $r < 1$, r^n becomes less as n increases,

i.e. $r^n \to 0 \quad \text{as} \quad n \to \infty$

Hence $\dfrac{ar^n}{(1-r)} \to 0 \quad \text{as} \quad n \to \infty$

Thus $S_n \to \dfrac{a}{(1-r)} \quad \text{as} \quad n \to \infty$

The quantity $\dfrac{a}{(1-r)}$ is called the **sum to infinity**, S_∞, and is the limiting value of the sum of an infinite number of terms,

i.e. $\boxed{\boldsymbol{S_\infty = \dfrac{a}{(1-r)}}}$ which is valid when $-1 < r < 1$

For example, the sum to infinity of the *GP* $1, \frac{1}{2}, \frac{1}{4}, \ldots$ is

$$S_\infty = \frac{1}{1 - \frac{1}{2}}, \text{ since } a = 1 \text{ and } r = \tfrac{1}{2}, \text{ i.e. } S_\infty = 2$$

29.7 Worked problems on geometric progressions

Problem 18. Determine the tenth term of the series 3, 6, 12, 24, ...

3, 6, 12, 24, ... is a geometric progression with a common ratio r of 2

The n'th term of a GP is ar^{n-1}, where a is the first term. Hence the 10th term is:

$$(3)(2)^{10-1} = (3)(2)^9 = 3(512) = \mathbf{1536}$$

Problem 19. Find the sum of the first 7 terms of the series, $\frac{1}{2}, 1\frac{1}{2}, 4\frac{1}{2}, 13\frac{1}{2}, \ldots$.

$\frac{1}{2}, 1\frac{1}{2}, 4\frac{1}{2}, 13\frac{1}{2}, \ldots$ is a GP with a common ratio $r = 3$

The sum of n terms, $S_n = \dfrac{a(r^n - 1)}{(r - 1)}$

Hence $S_7 = \dfrac{\frac{1}{2}(3^7 - 1)}{(3 - 1)} = \dfrac{\frac{1}{2}(2187 - 1)}{2} = \mathbf{546\frac{1}{2}}$

Problem 20. The first term of a geometric progression is 12 and the fifth term is 55. Determine the 8th term and the 11th term.

The 5th term is given by $ar^4 = 55$, where the first term $a = 12$

Hence $r^4 = \dfrac{55}{a} = \dfrac{55}{12}$ and $r = \sqrt[4]{\left(\dfrac{55}{12}\right)} = 1.4631719\ldots$.

The 8th term is

$$ar^7 = (12)(1.4631719\ldots)^7 = \mathbf{172.3}$$

The 11th term is

$$ar^{10} = (12)(1.4631719\ldots)^{10} = \mathbf{539.7}$$

Problem 21. Which term of the series 2187, 729, 243, ... is $\frac{1}{9}$?

2187, 729, 243, ... is a GP with a common ratio $r = \frac{1}{3}$ and first term $a = 2187$

The n'th term of a GP is given by: ar^{n-1}

Hence $\dfrac{1}{9} = (2187)\left(\dfrac{1}{3}\right)^{n-1}$ from which

$$\left(\frac{1}{3}\right)^{n-1} = \frac{1}{(9)(2187)} = \frac{1}{3^2 3^7} = \frac{1}{3^9} = \left(\frac{1}{3}\right)^9$$

Thus $(n - 1) = 9$, from which, $n = 9 + 1 = 10$

i.e. $\mathbf{\frac{1}{9}}$ **is the 10th term of the** ***GP***

Problem 22. Find the sum of the first 9 terms of the series 72.0, 57.6, 46.08, ...

The common ratio,

$$r = \frac{ar}{a} = \frac{57.6}{72.0} = 0.8 \left(\text{also } \frac{ar^2}{ar} = \frac{46.08}{57.6} = 0.8\right)$$

The sum of 9 terms,

$$S_9 = \frac{a(1 - r^n)}{(1 - r)} = \frac{72.0(1 - 0.8^9)}{(1 - 0.8)}$$
$$= \frac{72.0(1 - 0.1342)}{0.2} = \mathbf{311.7}$$

Problem 23. Find the sum to infinity of the series 3, 1, $\frac{1}{3}$, ...

3, 1, $\frac{1}{3}$, ... is a GP of common ratio, $r = \frac{1}{3}$

The sum to infinity,

$$S_\infty = \frac{a}{1 - r} = \frac{3}{1 - \frac{1}{3}} = \frac{3}{\frac{2}{3}} = \frac{9}{2} = \mathbf{4\frac{1}{2}}$$

Now try the following exercise

1. Find the 10th term of the series 5, 10, 20, 40, ...
2. Determine the sum of the first 7 terms of the series $\frac{1}{4}, \frac{3}{4}, 2\frac{1}{4}, 6\frac{3}{4}, \ldots$.
3. The first term of a geometric progression is 4 and the 6th term is 128. Determine the 8th and 11th terms.
4. Which term of the series 3, 9, 27, ... is 59049?
5. Find the sum of the first 7 terms of the series 2, 5, $12\frac{1}{2}$, ... (correct to 4 significant figures).
6. Determine the sum to infinity of the series 4, 2, 1, ...
7. Find the sum to infinity of the series $2\frac{1}{2}, -1\frac{1}{4}, \frac{5}{8}, \ldots$.

6.12 SEQUENCE OF SQUARE AND CUBES OF NATURAL NUMBERS

6.12.1. Natural Numbers

The positive integers 1, 2, 3, ... are known as natural numbers. Now we shall discuss the sum to n terms of some other special sequences (*i*) Sequence of natural numbers (*ii*) Sequence of square of natural numbers (*iii*) Sequence of cubes of natural numbers etc.

6.12.2. Sigma Notation

The sum of a series in short is written by placing the Greek better Σ (read as sigma) before the general term of the series.

Illustration : $\Sigma n = 1 + 2 + 3 + ... + n$

$$\Sigma n^2 = 1^2 + 2^2 + 3^2 + ... + n^2$$

$$\Sigma n^3 = 1^3 + 2^3 + 3^3 + ... + n^3.$$

$$\Sigma n(n+1) = 1.2 + 2.3 + ... + n(n+1)$$

$$\Sigma 3^n = 3^1 + 3^2 + 3^3 + ... + 3^n$$

$\Sigma a = na$, where a is constant independent of n.

6.12.3. To find the sum of first n natural numbers

Let S be the sum of first n natural numbers

$$\therefore \quad S = 1 + 2 + 3 + ... + n$$

This is an A.P. with $a = 1, l = n = \frac{n}{2}(1+n)$ $\left[\because S_n = \frac{n}{2}(a+l)\right]$

Hence $\Sigma n = \dfrac{n(n+1)}{2}$

6.12.4. To find the sum of first n odd natural numbers

Let S denote the sum of first n odd natural numbers

$$\therefore \quad S = 1 + 3 + 5 + ... + (2n-1) = \frac{n}{2}[1 + 2n - 1] = \frac{n}{2}(2n) = n^2$$

Hence $\Sigma(2n-1) = n^2$

6.12.5. To find the sum of squares of first n natural numbers

Let $S = 1^2 + 2^2 + 3^2 + ... + n^2 = \Sigma n^2.$

We know that for all values of x

$$x^3 - (x-1)^3 = 3x^2 - 3x + 1$$

Putting $\qquad x = 1, 2, 3, ..., n$, we get

$$\begin{aligned}
1^3 - 0^1 &= 3.1^2 - 3.1 + 1\\
2^3 - 1^3 &= 3.2^2 - 3.2 + 1\\
3^3 - 2^3 &= 3.3^2 - 3.3 + 1\\
&\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\\
n^3 - (n-1)^3 &= 3.n^2 - 3.n + 1
\end{aligned}$$

Adding, we get $n^3 - 0^3 = 3\,\Sigma n^2 - 3\Sigma n + n$

Hence $\qquad 3\Sigma n^2 = n^3 + 3\Sigma n - n = n^3 + 3\,\dfrac{n\,(n+1)}{2} - n = \dfrac{2n^3 + 3n^2 + 3n - 2n}{2}$

$$= \frac{2n^3 + 3n^2 + n}{2} = \frac{n\,(2n^2 + 3n + 1)}{2} = \frac{n\,(n+1)\,(2n+1)}{2}$$

$\therefore \qquad \Sigma n^2 = \dfrac{n\,(n+1)\,(2n+1)}{6}$

6.12.6. To find the sum of the cubes of first n natural numbers

Let S denote the sume of cubes of first n natural numbers.

$\therefore \qquad S = 1^3 + 2^3 + 3^3 + ... + n^3 = \Sigma n^3$,

We know that for all values of x; $x^4 - (x-1)^4 = 4x^3 - 6x^2 + 4x - 1$

Putting $\qquad x = 1, 2, 3, ... n$, we get

$$\begin{aligned}
1^4 - 0^4 &= 4.1^3 - 6.1^2 + 4.1 - 1\\
2^4 - 1^4 &= 4.2^3 - 6.2^2 + 4.2 - 1\\
3^4 - 2^4 &= 4.3^3 - 6.3^2 + 4.3 - 1\\
4^4 - 3^4 &= 4.4^3 - 6.4^2 + 4.4 - 1\\
&\cdots\cdots\cdots\cdots\\
n^4 - (n-1)^4 &= 4.n^3 - 6.n^2 + 4.n - 1
\end{aligned}$$

Adding, we get

$$\begin{aligned}
n^4 - 0^4 &= 4\Sigma n^3 - 6\Sigma n^2 + 4\Sigma n - n\\
\therefore \quad 4\Sigma n^3 &= n^4 + 6\Sigma n^6 - 4\Sigma n + n\\
&= n^4 + 6\,.\,\frac{n\,(n+1)\,(2n+1)}{6} - 4\,\frac{n\,(n+1)}{2} + n\\
&= n^4 + n\,(n+1)\,(2n+1) - 2n\,(n+1) + n = n\,[n^3 + 2n^2 + 3n + 1 - 2n - 2 + 1]\\
&= n\,[n^3 + 2n^2 + n] = n^2\,[n^2 + 2n + 1] = n^2\,(n+1)^2
\end{aligned}$$

$\therefore \qquad \Sigma n^3 = \dfrac{n^2\,(n+1)^2}{4} = \left[\dfrac{n\,(n+1)}{2}\right]^2 = (\Sigma n)^2$

Hence $\qquad \Sigma n^3 = \left[\dfrac{n\,(n+1)}{2}\right]^2 = (\Sigma n)^2$

Introduction to differentiation

33.1 Introduction to calculus

Calculus is a branch of mathematics involving or leading to calculations dealing with continuously varying functions.
Calculus is a subject that falls into two parts:

(i) **differential calculus** (or **differentiation**) and

(ii) **integral calculus** (or **integration**).

Differentiation is used in calculations involving rates of change (see section 33.10), velocity and acceleration, and maximum and minimum values of curves (see 'Engineering Mathematics').

33.2 Functional notation

In an equation such as $y=3x^2+2x-5$, y is said to be a function of x and may be written as $y=f(x)$.

An equation written in the form $f(x)=3x^2+2x-5$ is termed **functional notation**.

The value of $f(x)$ when $x=0$ is denoted by $f(0)$, and the value of $f(x)$ when $x=2$ is denoted by $f(2)$ and so on.

Thus when $f(x)=3x^2+2x-5$, then

$f(0)=3(0)^2+2(0)-5=-5$

and $f(2)=3(2)^2+2(2)-5=11$ and so on.

Problem 1. If $f(x)=4x^2-3x+2$ find: $f(0)$, $f(3)$, $f(-1)$ and $f(3)-f(-1)$

$$f(x)=4x^2-3x+2$$
$$f(0)=4(0)^2-3(0)+2=\mathbf{2}$$
$$f(3)=4(3)^2-3(3)+2=36-9+2=\mathbf{29}$$
$$f(-1)=4(-1)^2-3(-1)+2=4+3+2=\mathbf{9}$$
$$f(3)-f(-1)=29-9=\mathbf{20}$$

Problem 2. Given that $f(x)=5x^2+x-7$ determine:

(i) $f(2)\div f(1)$ (ii) $f(3+a)$

(iii) $f(3+a)-f(3)$ (iv) $\dfrac{f(3+a)-f(3)}{a}$

$$f(x)=5x^2+x-7$$

(i) $f(2)=5(2)^2+2-7=15$

$f(1)=5(1)^2+1-7=-1$

$f(2)\div f(1)=\dfrac{15}{-1}=\mathbf{-15}$

(ii) $f(3+a)=5(3+a)^2+(3+a)-7$

$=5(9+6a+a^2)+(3+a)-7$

$=45+30a+5a^2+3+a-7=\mathbf{41+31a+5a^2}$

(iii) $f(3)=5(3)^2+3-7=41$

$f(3+a)-f(3)=(41+31a+5a^2)-(41)=\mathbf{31a+5a^2}$

(iv) $\dfrac{f(3+a)-f(3)}{a}=\dfrac{31a+5a^2}{a}=\mathbf{31+5a}$

Now try the following exercise

1. If $f(x)=6x^2-2x+1$ find $f(0)$, $f(1)$, $f(2)$, $f(-1)$ and $f(-3)$
2. If $f(x)=2x^2+5x-7$ find $f(1)$, $f(2)$, $f(-1)$, $f(2)-f(-1)$

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33.5 Differentiation of $y = ax^n$ by the general rule

From differentiation by first principles, a general rule for differentiating ax^n emerges where a and n are any constants. This rule is:

$$\text{if} \quad \boxed{\begin{array}{l} \boldsymbol{y = ax^n} \quad \text{then} \quad \boldsymbol{\dfrac{dy}{dx} = anx^{n-1}} \\ \boldsymbol{f(x) = ax^n} \quad \text{then} \quad \boldsymbol{f'(x) = anx^{n-1}} \end{array}}$$

or, if

(Each of the results obtained in worked problems 3 and 7 may be deduced by using this general rule).

When differentiating, results can be expressed in a number of ways.

2

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For example: (i) if $y = 3x^2$ then $\dfrac{dy}{dx} = 6x$,
(ii) if $f(x) = 3x^2$ then $f'(x) = 6x$,
(iii) the differential coefficient of $3x^2$ is $6x$,
(iv) the derivative of $3x^2$ is $6x$, and
(v) $\dfrac{d}{dx}(3x^2) = 6x$

Problem 8. Using the general rule, differentiate the following with respect to x:

(a) $y = 5x^7$ (b) $y = 3\sqrt{x}$ (c) $y = \dfrac{4}{x^2}$

(a) Comparing $y = 5x^7$ with $y = ax^n$ shows that $a = 5$ and $n = 7$. Using the general rule,

$$\frac{\mathbf{dy}}{\mathbf{dx}} = \boldsymbol{anx^{n-1}} = (5)(7)x^{7-1} = \mathbf{35x^6}$$

(b) $y = 3\sqrt{x} = 3x^{\frac{1}{2}}$. Hence $a = 3$ and $n = \dfrac{1}{2}$

$$\frac{\mathbf{dy}}{\mathbf{dx}} = anx^{n-1} = (3)\frac{1}{2}x^{\frac{1}{2}-1}$$
$$= \frac{3}{2}x^{-\frac{1}{2}} = \frac{3}{2x^{\frac{1}{2}}} = \frac{\mathbf{3}}{\mathbf{2\sqrt{x}}}$$

(c) $y = \dfrac{4}{x^2} = 4x^{-2}$. Hence $a = 4$ and $n = -2$

$$\frac{\mathbf{dy}}{\mathbf{dx}} = anx^{n-1} = (4)(-2)x^{-2-1}$$
$$= -8x^{-3} = -\frac{\mathbf{8}}{\mathbf{x^3}}$$

Problem 9. Find the differential coefficient of $y = \dfrac{2}{5}x^3 - \dfrac{4}{x^3} + 4\sqrt{x^5} + 7$

$$y = \frac{2}{5}x^3 - \frac{4}{x^3} + 4\sqrt{x^5} + 7$$

i.e. $y = \dfrac{2}{5}x^3 - 4x^{-3} + 4x^{5/2} + 7$ from the laws of indices (see Chapter 3)

$$\frac{dy}{dx} = \left(\frac{2}{5}\right)(3)x^{3-1} - (4)(-3)x^{-3-1} + (4)\left(\frac{5}{2}\right)x^{(5/2)-1} + 0$$
$$= \frac{6}{5}x^2 + 12x^{-4} + 10x^{3/2}$$

i.e. $\dfrac{\mathbf{dy}}{\mathbf{dx}} = \dfrac{\mathbf{6}}{\mathbf{5}}\mathbf{x^2} + \dfrac{\mathbf{12}}{\mathbf{x^4}} + \mathbf{10\sqrt{x^3}}$

Problem 10. If $f(t) = 5t + \dfrac{1}{\sqrt{t^3}}$ find $f'(t)$

$$f(t) = 5t + \frac{1}{\sqrt{t^3}} = 5t + \frac{1}{t^{\frac{3}{2}}} = 5t^1 + t^{-\frac{3}{2}}$$

Hence $f'(t) = (5)(1)t^{1-1} + \left(-\dfrac{3}{2}\right)t^{-\frac{3}{2}-1} = 5t^0 - \dfrac{3}{2}t^{-\frac{5}{2}}$

i.e. $\boldsymbol{f'(t)} = 5 - \dfrac{3}{2t^{\frac{5}{2}}} = \mathbf{5} - \dfrac{\mathbf{3}}{\mathbf{2\sqrt{t^5}}}$ (since $t^0 = 1$)

Problem 11. Differentiate $y = \dfrac{(x+2)^2}{x}$ with respect to x

$$y = \frac{(x+2)^2}{x} = \frac{x^2 + 4x + 4}{x} = \frac{x^2}{x} + \frac{4x}{x} + \frac{4}{x}$$

i.e. $y = x^1 + 4 + 4x^{-1}$

Hence $\dfrac{\mathbf{dy}}{\mathbf{dx}} = 1x^{1-1} + 0 + (4)(-1)x^{-1-1} = x^0 - 4x^{-2}$

$$= \mathbf{1} - \frac{\mathbf{4}}{\mathbf{x^2}} \quad (\text{since } x^0 = 1)$$

Now try the following exercise

Exercise 121 Further problems on differentiation of $y = ax^n$ by the general rule (Answers on page 283)

In Problems 1 to 8, determine the differential coefficients with respect to the variable.

1. $y = 7x^4$
2. $y = \sqrt{x}$
3. $y = \sqrt{t^3}$
4. $y = 6 + \dfrac{1}{x^3}$
5. $y = 3x - \dfrac{1}{\sqrt{x}} + \dfrac{1}{x}$
6. $y = \dfrac{5}{x^2} - \dfrac{1}{\sqrt{x^7}} + 2$
7. $y = 3(t-2)^2$
8. $y = (x+1)^3$
9. Using the general rule for ax^n check the results of Problems 1 to 12 of Exercise 120, page 250.
10. Differentiate $f(x) = 6x^2 - 3x + 5$ and find the gradient of the curve at (a) $x = -1$, and (b) $x = 2$.
11. Find the differential coefficient of $y = 2x^3 + 3x^2 - 4x - 1$ and determine the gradient. of the curve at $x = 2$.
12. Determine the derivative of $y = -2x^3 + 4x + 7$ and determine the gradient of the curve at $x = -1.5$.

Introduction to integration

34.1 The process of integration

The process of integration reverses the process of differentiation. In differentiation, if $f(x)=2x^2$ then $f'(x)=4x$. Thus, the integral of $4x$ is $2x^2$, i.e. integration is the process of moving from $f'(x)$ to $f(x)$. By similar reasoning, the integral of $2t$ is t^2.

Integration is a process of summation or adding parts together and an elongated S, shown as $\int$, is used to replace the words 'the integral of'. Hence, from above, $\int 4x = 2x^2$ and $\int 2t$ is t^2.

In differentiation, the differential coefficient $\frac{dy}{dx}$ indicates that a function of x is being differentiated with respect to x, the dx indicating that it is 'with respect to x'. In integration the variable of integration is shown by adding d (the variable) after the function to be integrated.

Thus $\int 4x\, dx$ means 'the integral of $4x$ with respect to x', and $\int 2t\, dt$ means 'the integral of $2t$ with respect to t'

As stated above, the differential coefficient of $2x^2$ is $4x$, hence $\int 4x\, dx = 2x^2$. However, the differential coefficient of $2x^2+7$ is also $4x$. Hence $\int 4x\, dx$ could also be equal to $2x^2+7$. To allow for the possible presence of a constant, whenever the process of integration is performed, a constant 'c' is added to the result.

Thus $\int 4x\, dx = 2x^2+c$ and $\int 2t\, dt = t^2+c$

'c' is called the **arbitrary constant of integration.**

34.2 The general solution of integrals of the form ax^n

The general solution of integrals of the form $\int ax^n\, dx$, where a and n are constants and $n \neq -1$, is given by:

$$\boxed{\int \boldsymbol{ax^n}\, \mathbf{dx} = \frac{\boldsymbol{ax^{n+1}}}{\boldsymbol{n+1}} + \boldsymbol{c}}$$

Using this rule gives:

(i) $\displaystyle\int 3x^4\, dx = \frac{3x^{4+1}}{4+1} + c = \mathbf{\frac{3}{5}}\boldsymbol{x^5+c}$

(ii) $\displaystyle\int \frac{4}{9}t^3\, dt\, dx = \frac{4}{9}\left(\frac{t^{3+1}}{3+1}\right) + c$

$\displaystyle = \frac{4}{9}\left(\frac{t^4}{4}\right) + c = \mathbf{\frac{1}{9}}\boldsymbol{t^4+c}$

Both of these results may be checked by differentiation.

34.3 Standard integrals

From Chapter 33, $\frac{d}{dx}(\sin ax) = a\cos ax$

Since integration is the reverse process of differentiation it follows that:

$$\int a\cos ax\, dx = \sin ax + c$$

or $$\int \boldsymbol{\cos ax}\, \mathbf{dx} = \frac{\mathbf{1}}{\boldsymbol{a}}\boldsymbol{\sin ax + c}$$

By similar reasoning

$$\int \boldsymbol{\sin ax}\, \mathbf{dx} = -\frac{\mathbf{1}}{\boldsymbol{a}}\boldsymbol{\cos ax + c}$$

$$\int \boldsymbol{e^{ax}}\, \mathbf{dx} = \frac{\mathbf{1}}{\boldsymbol{a}}\boldsymbol{e^{ax} + c}$$

and $$\int \frac{\mathbf{1}}{\boldsymbol{x}}\mathbf{dx} = \boldsymbol{\ln x + c}$$

From above, $\int ax^n\, dx = \frac{ax^{n+1}}{n+1} + c$ **except when $n=-1$**

When $n=-1$, then $\int x^{-1}\, dx = \int \frac{1}{x}\, dx = \ln x + c$

A list of **standard integrals** is summarised in Table 34.1.

Table 34.1 Standard integrals

(i)	$\int ax^n \, dx$	$=$	$\dfrac{ax^{n+1}}{n+1} + c$ (except when $n = -1$)
(ii)	$\int \cos ax \, dx$	$=$	$\dfrac{1}{a} \sin ax + c$
(iii)	$\int \sin ax \, dx$	$=$	$-\dfrac{1}{a} \cos ax + c$
(iv)	$\int e^{ax} \, dx$	$=$	$\dfrac{1}{a} e^{ax} + c$
(v)	$\int \dfrac{1}{x} dx$	$=$	$\ln x + c$

Problem 1. Determine: (a) $\int 3x^2 \, dx$ (b) $\int 2t^3 \, dt$

The general rule is $\int ax^n \, dx = \dfrac{ax^{n+1}}{n+1} + c$

(a) When $a = 3$ and $n = 2$ then

$$\int 3x^2 \, dx = \frac{3x^{2+1}}{2+1} + c = \mathbf{x^3 + c}$$

(b) When $a = 2$ and $n = 3$ then

$$\int 2t^3 \, dt = \frac{2t^{3+1}}{3+1} + c = \frac{2t^4}{4} + c = \mathbf{\frac{1}{2}t^4 + c}$$

Each of these results may be checked by differentiating them.

Problem 2. Determine (a) $\int 8 \, dx$ (b) $\int 2x \, dx$

(a) $\int 8 \, dx$ is the same as $\int 8x^0 \, dx$ and, using the general rule when $a = 8$ and $n = 0$ gives:

$$\int 8x^0 \, dx = \frac{8x^{0+1}}{0+1} + c = \mathbf{8x + c}$$

In general, if k is a constant then $\int \mathbf{k \, dx = kx + c}$

(b) When $a = 2$ and $n = 1$, then

$$\int 2x \, dx = \int 2x^1 \, dx = \frac{2x^{1+1}}{1+1} + c = \frac{2x^2}{2} + c$$
$$= \mathbf{x^2 + c}$$

Problem 3. Determine: $\int \left(2 + \frac{5}{7}x - 6x^2\right) dx$

$\int \left(2 + \frac{5}{7}x - 6x^2\right) dx$ may be written as:

$$\int 2 \, dx + \int \frac{5}{7}x \, dx - \int 6x^2 \, dx$$

i.e. each term is integrated separately.
(This splitting up of terms only applies for addition and subtraction).

Hence $\int \left(2 + \frac{5}{7}x - 6x^2\right) dx$

$$= 2x + \left(\frac{5}{7}\right)\frac{x^{1+1}}{1+1} - (6)\frac{x^{2+1}}{2+1} + c$$
$$= 2x + \left(\frac{5}{7}\right)\frac{x^2}{2} - (6)\frac{x^3}{3} + c$$
$$= \mathbf{2x + \frac{5}{14}x^2 - 2x^3 + c}$$

Note that when an integral contains more than one term there is no need to have an arbitrary constant for each; just a single constant at the end is sufficient.

Problem 4. Determine: $\int \dfrac{3}{x^2} \, dx$

$\int \frac{3}{x^2} \, dx = \int 3x^{-2} \, dx$. Using the standard integral, $\int ax^n \, dx$ when $a = 3$ and $n = -2$ gives:

$$\int 3x^{-2} \, dx = \frac{3x^{-2+1}}{-2+1} + c$$
$$= \frac{3x^{-1}}{-1} + c = -3x^{-1} + c = \mathbf{\frac{-3}{x} + c}$$

Problem 5. Determine: $\int 3\sqrt{x} \, dx$

For fractional powers it is necessary to appreciate that $\sqrt[n]{a^m} = a^{\frac{m}{n}}$, from which, $\sqrt{x} = x^{\frac{1}{2}}$. Hence,

$$\int 3\sqrt{x} \, dx = \int 3x^{\frac{1}{2}} \, dx = \frac{3x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} + c$$
$$= (3)\left(\frac{2}{3}\right)x^{\frac{3}{2}} = 2x^{\frac{3}{2}} + c = \mathbf{2\sqrt{x^3} + c}$$

Problem 6. Determine: $\int \dfrac{5}{\sqrt{x}} \, dx$

$$\int \frac{5}{\sqrt{x}}\,dx = \int 5x^{-\frac{1}{2}}\,dx = \frac{5x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c = \frac{5x^{\frac{1}{2}}}{\frac{1}{2}} + c$$
$$= (5)\left(\frac{2}{1}\right)x^{\frac{1}{2}} + c = \mathbf{10\sqrt{x} + c}$$

Problem 7. Determine:
(a) $\int \left(\frac{x^3 - 2x}{3x}\right)dx$ (b) $\int (1-x)^2\,dx$

(a) Rearranging into standard integral form gives:

$$\int \left(\frac{x^3 - 2x}{3x}\right)dx = \int \left(\frac{x^3}{3x} - \frac{2x}{3x}\right)dx$$
$$= \int \left(\frac{x^2}{3} - \frac{2}{3}\right)dx$$
$$= \left(\frac{1}{3}\right)\frac{x^{2+1}}{2+1} - \frac{2}{3}x + c$$
$$= \left(\frac{1}{3}\right)\frac{x^3}{3} - \frac{2}{3}x + c$$
$$= \mathbf{\frac{1}{9}x^3 - \frac{2}{3}x + c}$$

(b) Rearranging $\int (1-x)^2\,dx$ gives:

$$\int (1 - 2x + x^2)\,dx = x - \frac{2x^{1+1}}{1+1} + \frac{x^{2+1}}{2+1} + c$$
$$= x - \frac{2x^2}{2} + \frac{x^3}{3} + c$$
$$= \mathbf{x - x^2 + \frac{1}{3}x^3 + c}$$

This problem shows that functions often have to be rearranged into the standard form of $\int ax^n\,dx$ before it is possible to integrate them.

Problem 8. Determine:
(a) $\int 5\cos 3x\,dx$ (b) $\int 3\sin 2x\,dx$

(a) From Table 34.1(ii),

$$\int 5\cos 3x\,dx = 5\int \cos 3x\,dx$$
$$= (5)\left(\frac{1}{3}\sin 3x\right) + c$$
$$= \mathbf{\frac{5}{3}\sin 3x + c}$$

(b) From Table 34.1(iii),

$$\int 3\sin 2x\,dx = 3\int \sin 2x\,dx = (3)\left(-\frac{1}{2}\cos 2x\right) + c$$
$$= \mathbf{-\frac{3}{2}\cos 2x + c}$$

Problem 9. Determine: (a) $\int 5e^{3x}\,dx$ (b) $\int \frac{6}{e^{2x}}\,dx$

(a) From Table 34.1(iv),

$$\int 5e^{3x}\,dx = 5\int e^{3x}\,dx = (5)\left(\frac{1}{3}e^{3x}\right) + c$$
$$= \mathbf{\frac{5}{3}e^{3x} + c}$$

(b) $$\int \frac{6}{e^{2x}}\,dx = \int 6e^{-2x}\,dx = 6\int e^{-2x}\,dx$$
$$= (6)\left(\frac{1}{-2}e^{-2x}\right) + c = -3e^{-2x} + c$$
$$= \mathbf{-\frac{3}{e^{2x}} + c}$$

Problem 10. Determine:
(a) $\int \frac{3}{5x}\,dx$ (b) $\int \left(\frac{3x^2 - 1}{x}\right)dx$

(a) $$\int \frac{3}{5x}\,dx = \int \left(\frac{3}{5}\right)\left(\frac{1}{x}\right)dx = \frac{3}{5}\int \left(\frac{1}{x}\right)dx$$
$$= \mathbf{\frac{3}{5}\ln x + c} \text{ (from Table 34.1(v))}$$

(b) $$\int \left(\frac{3x^2 - 1}{x}\right)dx = \int \left(\frac{3x^2}{x} - \frac{1}{x}\right)dx$$
$$= \int \left(3x - \frac{1}{x}\right)dx = \frac{3x^2}{2} + \ln x + c$$
$$= \mathbf{\frac{3}{2}x^2 + \ln x + c}$$

Now try the following exercise

Determine the following integrals:

1. (a) $\int 4\,dx$ (b) $\int 7x\,dx$
2. (a) $\int \frac{2}{5}x^2\,dx$ (b) $\int \frac{5}{6}x^3\,dx$

3. (a) $\int (3 + 2x - 4x^2)\,dx$ (b) $3\int (x + 5x^2)\,dx$

4. (a) $\int \left(\frac{3x^2 - 5x}{x}\right) dx$ (b) $\int (2 + x)^2\,dx$

5. (a) $\int \frac{4}{3x^2}\,dx$ (b) $\int \frac{3}{4x^4}\,dx$

6. (a) $\int \sqrt{x}\,dx$ (b) $\int \frac{2}{\sqrt{x}}\,dx$

7. (a) $\int 3\cos 2x\,dx$ (b) $\int 7\sin 3x\,dx$

8. (a) $\int \frac{3}{4}e^{2x}\,dx$ (b) $\frac{2}{3}\int \frac{dx}{e^{5x}}$

9. (a) $\int \frac{2}{3x}\,dx$ (b) $\int \left(\frac{x^2 - 1}{x}\right) dx$

MATRICES
AND DETERMINANTS

1.1 MATRIX ALGEBRA

Sir ARTHUR CAYLEY (1821-1895) of England was the first Mathematician to introduce the term MATRIX in the year 1858. But in the present day applied Mathematics in overwhelmingly large majority of cases it is used, as a notation to represent a large number of simultaneous equations in a compact and convenient manner.

Matrix Theory has its applications in Operations Research, Economics and Psychology. Apart from the above, matrices are now indispensible in all branches of Engineering, Physical and Social Sciences, Business Management, Statistics and Modern Control systems.

1.1.1 Definition of a Matrix

A rectangular array of numbers or functions represented by the symbol

$$\begin{pmatrix} a_{11} & a_{12} & \ldots a_{1n} \\ a_{21} & a_{22} & \ldots a_{2n} \\ . & . & . \\ . & . & . \\ . & . & . \\ a_{m1} & a_{m2} & \ldots a_{mn} \end{pmatrix} \text{ is called a MATRIX}$$

The numbers or functions a_{ij} of this array are called elements, may be real or complex numbers, where as m and n are positive integers, which denotes the number of Rows and number of Columns.

For example

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \text{ and } B = \begin{pmatrix} x^2 & \sin x \\ \sqrt{x} & \frac{1}{x} \end{pmatrix} \text{ are the matrices}$$

1.1.2 Order of a Matrix

A matrix A with m rows and n columns is said to be of the order m by n (m x n).

Symbolically

$A = (a_{ij})_{m \times n}$ is a matrix of order m x n. The first subscript i in (a_{ij}) ranging from 1 to m identifies the rows and the second subscript j in (a_{ij}) ranging from 1 to n identifies the columns.

For example

$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ is a Matrix of order 2 x 3 and

$B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ is a Matrix of order 2 x 2

$C = \begin{pmatrix} \sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{pmatrix}$ is a Matrix of order 2 x 2

$D = \begin{pmatrix} 0 & 22 & 30 \\ -4 & 5 & -67 \\ 78 & -8 & 93 \end{pmatrix}$ is a Matrix of order 3 x 3

1.1.3 Types of Matrices

(i) SQUARE MATRIX

When the number of rows is equal to the number of columns, the matrix is called a Square Matrix.

For example

$A = \begin{pmatrix} 5 & 7 \\ 6 & 3 \end{pmatrix}$ is a Square Matrix of order 2

$B = \begin{pmatrix} 3 & 1 & 5 \\ 4 & 1 & 6 \\ 2 & 4 & 9 \end{pmatrix}$ is a Square Matrix of order 3

$C = \begin{pmatrix} \sin\alpha & \sin\beta & \sin\delta \\ \cos\alpha & \cos\beta & \cos\delta \\ \mathrm{cosec}\,\alpha & \mathrm{cosec}\,\beta & \mathrm{cosec}\,\delta \end{pmatrix}$ is a Square Matrix of order 3

(ii) ROW MATRIX

A matrix having only one row is called Row Matrix

For example

A = $(2 \quad 0 \quad 1)$ is a row matrix of order 1 x 3

B = $(1 \quad 0)$ is a row matrix or order 1 x 2

(iii) COLUMN MATRIX

A matrix having only one column is called Column Matrix.

For example

$$A = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$ is a column matrix of order 3 x 1 and

$$B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$ is a column matrix of order 2 x 1

(iv) ZERO OR NULL MATRIX

A matrix in which all elements are equal to zero is called Zero or Null Matrix and is denoted by O.

For example

$$O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$ is a Null Matrix of order 2 x 2 and

$$O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ is a Null Matrix of order 2 x 3

(v) DIAGONAL MATRIX

A square Matrix in which all the elements other than main diagonal elements are zero is called a diagonal matrix

For example

$$A = \begin{pmatrix} 5 & 0 \\ 0 & 9 \end{pmatrix}$$ is a Diagonal Matrix of order 2 and

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$ is a Diagonal Matrix of order 3

Consider the square matrix

$$A = \begin{pmatrix} 1 & 3 & 7 \\ 5 & -2 & -4 \\ 3 & 6 & 5 \end{pmatrix}$$

Here 1, -2, 5 are called main diagonal elements and 3, -2, 7 are called secondary diagonal elements.

(vi) SCALAR MATRIX

A Diagonal Matrix with all diagonal elements equal to K (a scalar) is called a Scalar Matrix.

For example

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$ is a Scalar Matrix of order 3 and the value of scalar K = 2

(vii) UNIT MATRIX OR IDENTITY MATRIX

A scalar Matrix having each diagonal element equal to 1 (unity) is called a Unit Matrix and is denoted by I.

For example

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ is a Unit Matrix of order 2

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ is a Unit Matrix of order 3

1.1.4 Multiplication of a marix by a scalar

If $A = (a_{ij})$ is a matrix of any order and if K is a scalar, then the Scalar Multiplication of A by the scalar k is defined as

$$KA = (Ka_{ij}) \text{ for all } i, j.$$

In other words, to multiply a matrix A by a scalar K, multiply every element of A by K.

1.1.5 Negative of a matrix

The negative of a matrix $A = (a_{ij})_{m \times n}$ is defined by $-A = (-a_{ij})_{m \times n}$ for all i, j and is obtained by changing the sign of every element.

For example

$$\text{If } A = \begin{pmatrix} 2 & -5 & 7 \\ 0 & 5 & 6 \end{pmatrix} \text{then}$$

$$-A = \begin{pmatrix} -2 & 5 & -7 \\ 0 & -5 & -6 \end{pmatrix}$$

1.1.6 Equality of matrices

Two matrices are said to equal when

i) they have the same order and

ii) the corresponding elements are equal.

1.1.7 Addition of matrices

Addition of matrices is possible only when they are of same order (i.e., conformal for addition). When two matrices A and B are of same order, then their sum (A+B) is obtained by adding the corresponding elements in both the matrices.

1.1.8 Properties of matrix addition

Let A, B, C be matrices of the same order. The addition of matrices obeys the following

(i) Commutative law : $A + B = B + A$

(ii) Associative law : $A + (B + C) = (A + B) + C$

(iii) Distributive law : $K(A+B) = KA+KB$, where k is scalar.

1.1.9 Subtraction of matrices

Subtraction of matrices is also possible only when they are of same order. Let A and B be the two matrices of the same order. The matrix A - B is obtained by subtracting the elements of B from the corresponding elements of A.

1.1.10 Multiplication of matrices

Multiplication of two matrices is possible only when the number of columns of the first matrix is equal to the number of rows of the second matrix (i.e. conformable for multiplication)

Let $A = (a_{ij})$ be an m x p matrix,

and let $B = (b_{ij})$ be an p x n matrix.

Then the product AB is a matrix C = (c_{ij}) of order mxn,

where c_{ij} = element in the i^{th} row and j^{th} column of C is found by multiplying corresponding elements of the i^{th} row of A and j^{th} column of B and then adding the results.

For example

$$\text{if } A = \begin{pmatrix} 3 & 5 \\ 2 & -1 \\ 6 & 7 \end{pmatrix}_{3 \times 2} \quad B = \begin{pmatrix} 5 & -7 \\ -2 & 4 \end{pmatrix}_{2 \times 2}$$

$$\text{then } AB = \begin{pmatrix} 3 & 5 \\ 2 & -1 \\ 6 & 7 \end{pmatrix} \quad \begin{pmatrix} 5 & -7 \\ -2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \times 5 + 5\text{x}(-2) & 3\text{x}(-7) + 5\text{x}(5) \\ 2 \times 5 + (-1) \times (-2) & 2 \times (-7) + (-1) \times (4) \\ 6 \times 5 + 7\text{x}\ (-2) & 6\text{x}\ (-7) + 7\text{x}\ (4) \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ 12 & -18 \\ 16 & -14 \end{pmatrix}$$

1.1.11 Properties of matrix multiplication

(i) Matrix Multiplication is not commutative i.e. for the two matrices A and B, generally AB $\neq$ BA.

(ii) The Multiplication of Matrices is associative
i.e., (AB) C = A(BC)

(iii) Matrix Multiplication is distributive with respect to addition. i.e. if, A, B, C are matrices of order mxn, n x k, and n x k respectively, then A(B+C) = AB + AC

(iv) Let A be a square matrix of order n and I is the unit matrix of same order.
Then AI = A = I A

(v) The product
AB = O (Null matrix), does not imply that either A = 0 or B = 0 or both are zero.

For example

$$\text{Let } A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}_{2\times 2} \quad B = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}_{2\times 2}$$

$$\text{Then } AB = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow$ AB = (null matrix)

Here neither the matrix A, nor the matrix B is Zero, but the product AB is zero.

1.1.12 Transpose of a matrix

Let A = (a_{ij}) be a matrix of order mxn. The transpose of A, denoted by A^T of order nxm is obtained by interchanging rows into columns of A.

For example

$$\text{If} \quad A = \begin{pmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{pmatrix}_{2\times 3}, \text{ then}$$

$$A^T = \begin{pmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 6 \end{pmatrix}$$

1.1.13 Properties Of Matrix Transposition

Let A^T and B^T are the transposed Matrices of A and B and α is a scalar. Then

(i) $(A^T)^T = A$

(ii) $(A + B)^T = A^T + B^T$

(iii) $(\alpha A)^T = \alpha A^T$

(iv) $(AB)^T = B^T A^T$ (A and B are conformable for multiplication)

Example 1

$$\textbf{If } A = \begin{pmatrix} 5 & 9 & 6 \\ 6 & 2 & 10 \end{pmatrix} \textbf{ and } B = \begin{pmatrix} 6 & 0 & 7 \\ 4 & -8 & -3 \end{pmatrix}$$

find A + B and A-B

Solution :

$$A+B = \begin{pmatrix} 5+6 & 9+0 & 6+7 \\ 6+4 & 2+(-8) & 10+(-3) \end{pmatrix} = \begin{pmatrix} 11 & 9 & 13 \\ 10 & -6 & 7 \end{pmatrix}$$

$$A\text{-}B = \begin{pmatrix} 5-6 & 9-0 & 6-7 \\ 6-4 & 2-(-8) & 10-(-3) \end{pmatrix} = \begin{pmatrix} -1 & 9 & -1 \\ 2 & 10 & 13 \end{pmatrix}$$

Example 2

If $A = \begin{pmatrix} 3 & 6 \\ 9 & 2 \end{pmatrix}$ **find (i) 3A (ii)** $-\frac{1}{3}$ **A**

Solution :

$$\text{(i) } 3A = 3\begin{pmatrix} 3 & 6 \\ 9 & 2 \end{pmatrix} = \begin{pmatrix} 9 & 18 \\ 27 & 6 \end{pmatrix}$$

$$\text{(ii) } -\frac{1}{3}\ A = -\frac{1}{3}\begin{pmatrix} 3 & 6 \\ 9 & 2 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ -3 & -\frac{2}{3} \end{pmatrix}$$

Example 3

If $A = \begin{pmatrix} 2 & 3 & 5 \\ 4 & 7 & 9 \\ 1 & 6 & 4 \end{pmatrix}$ **and** $B = \begin{pmatrix} 3 & 1 & 2 \\ 4 & 2 & 5 \\ 6 & -2 & 7 \end{pmatrix}$

show that 5(A+B) = 5A + 5B

Solution :

$$A+B = \begin{pmatrix} 5 & 4 & 7 \\ 8 & 9 & 14 \\ 7 & 4 & 11 \end{pmatrix} \quad \therefore\ 5(A+B) = \begin{pmatrix} 25 & 20 & 35 \\ 40 & 45 & 70 \\ 35 & 20 & 55 \end{pmatrix}$$

$$5A = \begin{pmatrix} 10 & 15 & 25 \\ 20 & 35 & 45 \\ 5 & 30 & 20 \end{pmatrix} \text{ and } 5B = \begin{pmatrix} 15 & 5 & 10 \\ 20 & 10 & 25 \\ 30 & -10 & 35 \end{pmatrix}$$

$$\therefore\ 5A+5B = \begin{pmatrix} 25 & 20 & 35 \\ 40 & 45 & 70 \\ 35 & 20 & 55 \end{pmatrix} \quad \therefore\ 5(A+B) = 5A + 5B$$

Example 4

If A = $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$ and B = $\begin{pmatrix} -1 & -2 & -4 \\ -1 & -2 & -4 \\ 1 & 2 & 4 \end{pmatrix}$

find AB and BA. Also show that AB ¹ BA

Solution:

$$AB = \begin{pmatrix} 1(-1)+2(-1)+3(1) & 1(-2)+2(-2)+3x2 & 1(-4)+2(-4)+3x4 \\ 2(-1)+4(-1)+6(1) & 2(-2)+4(-2)+6(2) & 2(-4)+4(-4)+6x4 \\ 3(-1)+6(-1)+9(1) & 3(-2)+6(-2)+9(2) & 3(-4)+6(-4)+9x4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{3x\,3}$$

$$\text{Similarly BA} = \begin{pmatrix} -17 & -34 & -51 \\ -17 & -34 & -51 \\ 17 & 34 & 51 \end{pmatrix}$$

$$\therefore AB \neq BA$$

Example 5

If A = $\begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}$, then compute A^2-5A + 3I

Solution:

$$A^2 = A.A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} = \begin{pmatrix} -5 & 6 \\ -9 & 10 \end{pmatrix}$$

$$5A = 5\begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} = \begin{pmatrix} 5 & -10 \\ 15 & -20 \end{pmatrix}$$

$$3I = 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\therefore A^2 - 5A+3I = \begin{pmatrix} -5 & 6 \\ -9 & 10 \end{pmatrix} - \begin{pmatrix} 5 & -10 \\ 15 & -20 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -10 & 16 \\ -24 & 30 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} -7 & 16 \\ -24 & 33 \end{pmatrix}$$

Example 6

Verify that $(AB)^T = B^T A^T$ when

$$\mathbf{A} = \begin{pmatrix} 1 & -4 & 2 \\ 4 & 0 & 1 \end{pmatrix}_{2\times 3} \quad \textbf{and} \quad \mathbf{B} = \begin{pmatrix} 2 & -3 \\ 0 & 1 \\ -4 & -2 \end{pmatrix}_{3\times 2}$$

Solution :

$$AB = \begin{pmatrix} 1 & -4 & 2 \\ 4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 0 & 1 \\ -4 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1x2 + (-4)x0 + 2(-4) & 1x(-3) + (-4)x1 + 2x(-2) \\ 4x2 + 0x0 + 1x(-4) & 4x(-3) + 0x1 + 1x(-2) \end{pmatrix}$$

$$= \begin{pmatrix} 2+0-8 & -3-4-4 \\ 8+0-4 & -12+0-2 \end{pmatrix} = \begin{pmatrix} -6 & -11 \\ 4 & -14 \end{pmatrix}$$

$$\therefore \text{L.H.S.} = (AB)^T = \begin{pmatrix} -6 & -11 \\ 4 & -14 \end{pmatrix}^T = \begin{pmatrix} -6 & 4 \\ -11 & -14 \end{pmatrix}$$

$$\text{R.H.S.} = B^T A^T = \begin{pmatrix} 2 & 0 & -4 \\ -3 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ -4 & 0 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -6 & 4 \\ -11 & -14 \end{pmatrix}$$

$\Rightarrow$ L.H.S. = R.H.S

Example 7

A radio manufacturing company produces three models of radios say A, B and C. There is an export order of 500 for model A, 1000 for model B, and 200 for model C. The material and labour (in appropriate units) needed to produce each model is given by the following table:

$$\begin{array}{l} \\ \textbf{Model A} \\ \textbf{Model B} \\ \textbf{Model C} \end{array} \begin{array}{c} \begin{array}{cc} \text{Material} & \text{Labour} \end{array} \\ \begin{pmatrix} \mathbf{10} & \mathbf{20} \\ \mathbf{8} & \mathbf{5} \\ \mathbf{12} & \mathbf{9} \end{pmatrix} \end{array}$$

Use marix multiplication to compute the total amount of material and labour needed to fill the entire export order.

Solution:

Let P denote the matrix expressing material and labour corresponding to the models A, B, C. Then

$$P = \begin{array}{c} \begin{matrix} \text{Material} & \text{Labour} \end{matrix} \\ \begin{pmatrix} 10 & 20 \\ 8 & 05 \\ 12 & 9 \end{pmatrix} \end{array} \begin{matrix} \text{Model A} \\ \text{Model B} \\ \text{Model C} \end{matrix}$$

Let E denote matrix expressing the number of units ordered for export in respect of models A, B, C. Then

$$E = \begin{array}{c} \begin{matrix} A & B & C \end{matrix} \\ (500 \;\; 1000 \;\; 200) \end{array}$$

$\therefore$ Total amount of material and labour $= E \times P$

$$= (500 \;\; 1000 \;\; 200) \begin{pmatrix} 10 & 20 \\ 8 & 5 \\ 12 & 9 \end{pmatrix}$$

$$= (5000 + 8000 + 2400 \qquad 10000 + 5000 + 1800)$$

$$= \begin{array}{c} \begin{matrix} \text{Material} & \text{Labour} \end{matrix} \\ (15{,}400 \qquad 16{,}800) \end{array}$$

Example 8

Two shops A and B have in stock the following brand of tubelights

Shops	Brand		
	Bajaj	**Philips**	**Surya**
Shop A	**43**	**62**	**36**
Shop B	**24**	**18**	**60**

Shop A places order for 30 Bajaj, 30 Philips, and 20 Surya brand of tubelights, whereas shop B orders 10, 6, 40 numbers of the three varieties. Due to the various factors, they receive only half of the order as supplied by the manufacturers. The cost of each tubelights of the three types are Rs. 42, Rs. 38 and Rs. 36 respectively. Represent the following as matrices (i) Initial stock (ii) the order (iii) the supply (iv) final sotck (v) cost of individual items (column matrix) (vi) total cost of stock in the shops.

Solution:

(i) The initial stock matrix $P = \begin{pmatrix} 43 & 62 & 36 \\ 24 & 18 & 60 \end{pmatrix}$

(ii) The order matrix $Q = \begin{pmatrix} 30 & 30 & 20 \\ 10 & 6 & 40 \end{pmatrix}$

(iii) The supply matrix $R = \frac{1}{2}Q = \begin{pmatrix} 15 & 15 & 10 \\ 5 & 3 & 20 \end{pmatrix}$

(iv) The final stock matrix $S = P + R = \begin{pmatrix} 58 & 77 & 46 \\ 29 & 21 & 80 \end{pmatrix}$

(v) The cost vector $C = \begin{pmatrix} 42 \\ 38 \\ 36 \end{pmatrix}$

(vi) The total cost stock in the shops

$$T = SC = \begin{pmatrix} 58 & 77 & 46 \\ 29 & 21 & 80 \end{pmatrix}\begin{pmatrix} 42 \\ 38 \\ 36 \end{pmatrix}$$

$$= \begin{pmatrix} 2436 + 2926 + 1656 \\ 1218 + 798 + 2880 \end{pmatrix} = \begin{pmatrix} 7018 \\ 4896 \end{pmatrix}$$

EXERCISE 1.1

1) If $A = \begin{pmatrix} 5 & 3 \\ 7 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 2 \\ 4 & 6 \end{pmatrix}$ then, show that

(i) $A + B = B + A$ (ii) $(A^T)^T = A$

2. If $A = \begin{pmatrix} 3 & 1 & 2 \\ 4 & 9 & 8 \\ 2 & 5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 9 & 2 & 5 \\ 0 & 3 & -1 \\ 4 & -6 & 2 \end{pmatrix}$

find (i) A + B (iii) 5A and 2B

(ii) B + A (iv) 5A + 2B

3) If $A = \begin{pmatrix} 1 & -2 \\ 3 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 4 \\ -3 & 0 \end{pmatrix}$, find AB and BA.

4) Find AB and BA when

$$A = \begin{pmatrix} -3 & 1 & -5 \\ -1 & 5 & 2 \\ -2 & 4 & -3 \end{pmatrix} \text{ and } B = \begin{pmatrix} -2 & 4 & 5 \\ 0 & 2 & 1 \\ -1 & 6 & 3 \end{pmatrix}$$

5) If $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 5 \\ 7 & 3 \\ 5 & -2 \end{pmatrix}$, find AB and BA.

6) If $A = \begin{pmatrix} 3 & 4 \\ 1 & 1 \\ 2 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & 1 \\ 3 & -2 \end{pmatrix}$

verify that $(AB)^T = B^T A^T$

7) Let $A = \begin{pmatrix} 2 & -1 & 4 \\ 3 & 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -5 \end{pmatrix}$ then

show that 3 (A+B) = 3A + 3B.

8) If $A = \begin{pmatrix} 12 & 11 \\ 9 & -7 \end{pmatrix}$, $\alpha = 3, \beta = -7$,

show that $(\alpha + \beta)A = \alpha A + \beta A$.

9) Verify that $\alpha (A + B) = \alpha A + \alpha B$ where

$$\alpha = 3, \quad A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 0 & 2 \\ 4 & 3 & 5 \end{pmatrix} \text{ and } B = \begin{pmatrix} 5 & 3 & -1 \\ 7 & 2 & 4 \\ 3 & 1 & 2 \end{pmatrix}$$

10) If $A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ and $B = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$

prove that (i) AB = BA (ii) $(A+B)^2 = A^2 + B^2 + 2AB$.

11) If $A = (3\ \ 5\ \ 6)_{1 \times 3}$, and $B = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}_{3 \times 1}$ then find AB and BA.

12) If A = $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ and B = $\begin{pmatrix} \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} \end{pmatrix}$ find AB, BA

13) There are two families A and B. There are 4 men, 2 women and 1 child in family A and 2 men, 3 women and 2 children in family B. They recommended daily allowance for calories i.e. Men : 2000, Women : 1500, Children : 1200 and for proteins is Men : 50 gms., Women : 45 gms., Children : 30 gms.

Represent the above information by matrices using matrix multiplication, calculate the total requirements of calories and proteins for each of the families.

14) Find the sum of the following matrices

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 7 & 10 & 12 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 2 & 2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 8 & 9 & 7 \\ 7 & 8 & 6 \\ 9 & 10 & 8 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 7 & 13 & 19 \end{pmatrix}$$

15) If x + $\begin{pmatrix} 5 & 6 \\ 7 & 0 \end{pmatrix}$ = $2I_2 + 0_2$ then find x

16) If A = $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ show that (A - I) (A - 4I) = 0

17) If A = $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ and B = $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ then show that

(i) $(A+B)(A-B) \neq A^2 - B^2$ (ii) $(A+B)^2 \neq A^2 + 2AB + B^2$

18) If 3A + $\begin{pmatrix} 4 & -1 \\ -2 & 1 \end{pmatrix}$ = $\begin{pmatrix} -2 & 2 \\ 1 & 4 \end{pmatrix}$, find the value of A

19) Show that A = $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ satisfies $A^2 = -I$

20) If $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ prove that $A^2 = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$

21) If $A = \begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix}$ show that A^2, A^4 are identity matrices

22) If $A = \begin{pmatrix} 7 & 1 \\ 0 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 2 \\ 4 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 5 & 2 \\ -1 & 3 \end{pmatrix}$

Evluate (i) (A+B) (C+D) (ii) (C+D) (A+B) (iii) $A^2 - B^2$ (iv) $C^2 + D^2$

23) The number of students studying Business Mathematics, Economics, Computer Science and Statistics in a school are given below

Std.	Business Mathematics	Economics	Computer Science	Statistics
XI Std.	45	60	55	30
XII Std.	58	72	40	80

(i) Express the above data in the form of a matrix

(ii) Write the order of the matrix

(iii) Express standardwise the number of students as a column matrix and subjectwise as a row matrix.

(iv) What is the relationship between (i) and (iii)?

1.2 DETERMINANTS

An important attribute in the study of Matrix Algebra is the concept of **Determinant,** ascribed to a square matrix. A knowledge of **Determinant** theory is indispensable in the study of Matrix Algebra.

1.2.1 Determinant

The determinant associated with each square matrix $A = (a_{ij})$ is a **scalar** and denoted by the symbol det.A or $|A|$. The scalar may be real or complex number, positive, Negative or Zero. A matrix is an array and has **no numerical** value, but a determinant has **numerical value.**

For example

when $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then determinant of A is

$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ and the determinant value is = ad - bc

Example 9

Evaluate $\begin{vmatrix} \mathbf{1} & \mathbf{-1} \\ \mathbf{3} & \mathbf{-2} \end{vmatrix}$

Solution:

$$\begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix}$$

= 1 x (-2) - 3 x (-1) = -2 + 3 = 1

Example 10

Evaluate $\begin{vmatrix} \mathbf{2} & \mathbf{0} & \mathbf{4} \\ \mathbf{5} & \mathbf{-1} & \mathbf{1} \\ \mathbf{9} & \mathbf{7} & \mathbf{8} \end{vmatrix}$

Solution:

$$\begin{vmatrix} 2 & 0 & 4 \\ 5 & -1 & 1 \\ 9 & 7 & 8 \end{vmatrix} = 2\begin{vmatrix} -1 & 1 \\ 7 & 8 \end{vmatrix} - 0\begin{vmatrix} 5 & 1 \\ 9 & 8 \end{vmatrix} + 4\begin{vmatrix} 5 & -1 \\ 9 & 7 \end{vmatrix}$$

= 2 (-1 x 8 - 1 x 7) - 0 (5 x 8 -9 x 1) + 4 (5x7 - (-1) x 9)

= 2 (-8 -7) - 0 (40 - 9) + 4 (35 + 9)

= -30 - 0 + 176 = 146

1.2.2 Properties Of Determinants

(i) The value of determinant is unaltered, when its rows and columns are interchanged.

(ii) If any two rows (columns) of a determinant are interchanged, then the value of the determinant changes only in sign.

(iii) If the determinant has two identical rows (columns), then the value of the determinant is zero.

(iv) If all the elements in a row or in a (column) of a determinant are multiplied by a constant k(k, $\neq$ 0) then the value of the determinant is multiplied by k.

(v) The value of the determinant is unaltered when a constant multiple of the elements of any row (column), is added to the corresponding elements of a different row (column) in a determinant.

(vi) If each element of a row (column) of a determinant is expressed as the sum of two or more terms, then the determinant is expressed as the sum of two or more determinants of the same order.

(vii) If any two rows or columns of a determinant are proportional, then the value of the determinant is zero.

1.2.3 Product of Determinants

Product of two determinants is possible only when they are of the same order. Also $|AB| = |A| \cdot |B|$

Example 11

Evaluate $|A|\,|B|$, if $A = \begin{vmatrix} 3 & 1 \\ 5 & 6 \end{vmatrix}$ **and** $B = \begin{vmatrix} 5 & 2 \\ 1 & 3 \end{vmatrix}$

Solution:

Multiplying row by column

$$|A|\;|B| = \begin{vmatrix} 3 & 1 \\ 5 & 6 \end{vmatrix} \begin{vmatrix} 5 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 3 \text{ x } 5 + 1 \text{ x } 1 & 3 \text{ x } 2 + 1 \text{ x } 3 \\ 5 \text{ x } 5 + 6 \text{ x } 1 & 5 \text{ x } 2 + 6 \text{ x } 3 \end{vmatrix}$$

$$= \begin{vmatrix} 15+1 & 6+3 \\ 25+6 & 10+18 \end{vmatrix} = \begin{vmatrix} 16 & 9 \\ 31 & 28 \end{vmatrix} = 448 - 279$$

$$= 169$$

Example 12

Find $\begin{vmatrix} 2 & 1 & 3 \\ 3 & 0 & 5 \\ 1 & 0 & -4 \end{vmatrix} \begin{vmatrix} 2 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 2 & 0 \end{vmatrix}$

Solution :

Multiplying row by column

$$\begin{vmatrix} 2 & 1 & 3 \\ 3 & 0 & 5 \\ 1 & 0 & -4 \end{vmatrix} \begin{vmatrix} 2 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 2 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 2 \text{ x } 2 + 1 \text{ x } 0 + 3 \text{ x } 0 & 2 \text{ x } 0 + 1 \text{ x } 0 + 3 \text{ x } 2 & 2 \text{ x } 0 + 1 \text{ x } 3 + 3 \text{ x } 0 \\ 3 \text{ x } 2 + 0 \text{ x } 0 + 5 \text{ x } 0 & 3 \text{ x } 0 + 0 \text{ x } 0 + 5 \text{ x } 2 & 3 \text{ x } 0 + 0 \text{ x } 3 + 5 \text{ x } 0 \\ 1 \text{ x } 2 + 0 \text{ x } 0 - 4 \text{ x } 0 & 1 \text{ x } 0 + 0 \text{ x } 0 - 4 \text{ x } 2 & 1 \text{ x } 0 + 0 \text{ x } 3 - 4 \text{ x } 0 \end{vmatrix}$$

$$= \begin{vmatrix} 4 & 6 & 3 \\ 6 & 10 & 0 \\ 2 & -8 & 0 \end{vmatrix}$$

$= 4\ (0 + 0)\ -6\ (0 - 0) + 3\ (-48 - 20)$

$= 3\ (-68) = -\ 204$

1.2.4 Singular Matrix

A square matrix A is said to be singular if det. A = 0, otherwise it is a non-singular matrix.

Example 13

Show that $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ is a singular matrix

Solution:

$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 4 - 4 = 0$

$\therefore$ The matrix is singular

Example 14

Show that $\begin{pmatrix} 2 & 5 \\ 9 & 10 \end{pmatrix}$ is a non-singular matrix

Solution :

$\begin{vmatrix} 2 & 5 \\ 9 & 10 \end{vmatrix} = 29 - 45 = -25 \neq 0$

$\therefore$ The given matrix is non singular

Example : 15

Find x if $\begin{vmatrix} \mathbf{1} & \mathbf{x} & \mathbf{-4} \\ \mathbf{5} & \mathbf{3} & \mathbf{0} \\ \mathbf{-2} & \mathbf{-4} & \mathbf{8} \end{vmatrix}$ **= 0**

Solution :

Expanding by 1st Row,

$$\begin{vmatrix} 1 & x & -4 \\ 5 & 3 & 0 \\ -2 & -4 & 8 \end{vmatrix} = 1\begin{vmatrix} 3 & 0 \\ -4 & 8 \end{vmatrix} - x\begin{vmatrix} 5 & 0 \\ -2 & 8 \end{vmatrix} + (-4)\begin{vmatrix} 5 & 3 \\ -2 & -4 \end{vmatrix}$$

$= \quad 1(24) - x\,(40) -4\,(-20 +6)$

$= \quad 24 -40x + 56 = -40x + 80$

$\Rightarrow \quad -40\,x + 80 = 0$

$\therefore \quad x = 2$

Example : 16

Show $\begin{vmatrix} \mathbf{1} & \mathbf{b+c} & \mathbf{b^2+c^2} \\ \mathbf{1} & \mathbf{c+a} & \mathbf{c^2+a^2} \\ \mathbf{1} & \mathbf{a+b} & \mathbf{a^2+b^2} \end{vmatrix}$ **= (a-b) (b-c) (c-a)**

Solution :

$$\begin{vmatrix} 1 & b+c & b^2+c^2 \\ 1 & c+a & c^2+a^2 \\ 1 & a+b & a^2+b^2 \end{vmatrix}$$

$R_2 \to R_2 - R_1, \quad R_3 \to R_3 - R_1$

$$= \begin{vmatrix} 1 & b+c & b^2+c^2 \\ 0 & a-b & a^2+b^2 \\ 0 & a-c & a^2-c^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & b+c & b^2+c^2 \\ 0 & a-b & (a+b)(a-b) \\ 0 & a-c & (a+c)(a-c) \end{vmatrix}$$ taking out (a-b) from R_2 and (a-c) from R_3

$$= (a-b)\,(a-c)\begin{vmatrix} 1 & b+c & b^2+c^2 \\ 0 & 1 & a+b \\ 0 & 1 & a+c \end{vmatrix}$$

$= (a-b)\,(a-c)\,[a+c-a-b]$ (Expanding along c_1)

$= (a-b)\,(a-c)\,(c-b) \; = \; (a-b)\,(b-c)\,(c-a)$

EXERCISE 1.2

1) Evaluate (i) $\begin{vmatrix} 4 & 6 \\ -2 & 3 \end{vmatrix}$ (ii) $\begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix}$ (iii) $\begin{vmatrix} -2 & -4 \\ -1 & -6 \end{vmatrix}$

2) Evaluate $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \\ 1 & 2 & 4 \end{vmatrix}$ 3) Evaluate $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

4) Examine whether $A = \begin{pmatrix} 7 & 4 & 3 \\ 3 & 2 & 1 \\ 5 & 3 & 2 \end{pmatrix}$ is non-singular

5) Examine whether the given matrix $A = \begin{pmatrix} 1 & -2 & 3 \\ -2 & -1 & 0 \\ 4 & -2 & 5 \end{pmatrix}$ is singular

6) Evaluate $\begin{vmatrix} 3 & 2 & 1 \\ 0 & 1 & 4 \\ 3 & 2 & 1 \end{vmatrix}$ 7) Evaluate $\begin{vmatrix} 1 & 4 & 2 \\ 2 & -2 & 4 \\ 3 & -1 & 6 \end{vmatrix}$

8) If the value of $\begin{vmatrix} 2 & 3 & 5 \\ 4 & 1 & 0 \\ 6 & 2 & 7 \end{vmatrix} = -60$, then evaluate $\begin{vmatrix} 2 & 6 & 5 \\ 4 & 2 & 0 \\ 6 & 4 & 7 \end{vmatrix}$

9) If the value of $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \\ 2 & 0 & 1 \end{vmatrix} = 5$, then what is the value of $\begin{vmatrix} 1 & 8 & 3 \\ 1 & 7 & 3 \\ 2 & 12 & 1 \end{vmatrix}$

10) Show that $\begin{vmatrix} 2+4 & 6+3 \\ 1 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 6 \\ 1 & 5 \end{vmatrix} + \begin{vmatrix} 4 & 3 \\ 1 & 5 \end{vmatrix}$

11) Prove that $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

12) Prove that $\begin{vmatrix} b+c & a & 1 \\ c+a & b & 1 \\ a+b & c & 1 \end{vmatrix} = 0$

13) Show that $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix} = xy$

EXERCISE 1.3

Choose the correct answer

1) [0 0 0] is a

(a) Unit matrix (b) Scalar matrix
(c) Null matrix (d) Diagonal matrix

2) [6 2 - 3] is a matrix of order

(a) 3 x 3 (b) 3 x 1
(c) 1 x 3 (d) Scalar matrix

3) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is a

(a) Unit matrix (b) Zero matrix of order 2 x 2
(c) Unit matrix of 2 x 2 (c) None of these

4) $A = \begin{pmatrix} 3 & -3 \\ 2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$, then A + B is

(a) $\begin{pmatrix} 4 & 5 \\ 3 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}$

(c) $\begin{pmatrix} 4 & -1 \\ 1 & 4 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

5) If A = $\begin{pmatrix} 8 & 9 \\ -3 & -1 \end{pmatrix}$ and B = $\begin{pmatrix} -1 & 3 \\ 0 & -2 \end{pmatrix}$, then A - B is

(a) $\begin{pmatrix} 7 & 6 \\ -3 & -3 \end{pmatrix}$ (b) $\begin{pmatrix} 9 & 6 \\ -3 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 7 & 6 \\ 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

6) If A = $\begin{pmatrix} 2 & 4 \\ -3 & -3 \end{pmatrix}$, then -3A is

(a) $\begin{pmatrix} -6 & -12 \\ -9 & 15 \end{pmatrix}$ (b) $\begin{pmatrix} -6 & -12 \\ 9 & 15 \end{pmatrix}$ (c) $\begin{pmatrix} -6 & 12 \\ 9 & 9 \end{pmatrix}$ (d) None of these

7) If A = $\begin{pmatrix} 2 & 3 & 4 \\ 1 & -1 & 0 \\ 5 & -3 & 1 \end{pmatrix}$ and I = $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, then A + 2I is

(a) $\begin{pmatrix} 4 & 3 & 4 \\ 1 & 1 & 0 \\ 5 & -3 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 3 & 4 \\ 1 & 0 & 0 \\ 5 & -3 & 2 \end{pmatrix}$

(c) $\begin{pmatrix} 4 & 3 & 4 \\ 1 & -1 & 0 \\ 5 & -3 & 2 \end{pmatrix}$ (d) None of these

8) $\begin{pmatrix} 3 & 5 & 6 \\ -2 & 1 & 6 \end{pmatrix}$ x $\begin{pmatrix} 5 & -1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$

a) $\begin{pmatrix} 15 & 12 \\ -4 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} -3 & 15 \\ 8 & -3 \end{pmatrix}$

(c) Cannot be multiplied (d) None of these

9) The value of $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ is

(a) 4 (b) 14 (c) -14 (d) None of these

10) The value of $\begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix}$ is

(a) 0 (b) - 1 (c) 1 (d) None of these

11) If the value of $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2$, then the value of $\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}$ is

(a) 0 (b) - 2 (c) 2 (d) None of these

12) Det (AB) = | AB | = ?

(a) $|A| + |B|$ (b) $|B| + |A|$

(c) $|A| \times |B|$ (d) None of these

13) The element at 2nd Row and 2nd Coloumn is denoted by

(a) a_{12} (b) a_{32} (c) a_{22} (d) a_{11}

14) Order of the matrix $A = [a_{ij}]_{3 \times 3}$ is

(a) 2 x 3 (b) 3 x 3 (c) 1 x 3 (d) 3 x 1

15) When the number of rows and the number of coloumns of a matrix are equal, the matrix is

(a) square matrix (b) row matrix (c) column matrix (d) None of these

16) If all the elements of a matrix are zeros, then the matrix is a

(a) unit matrix (b) square matrix

(c) zero matrix (d) None of these

17) A diagonal matrix in which all the diagonal elements are equal is a

(a) scalar matrix (b) column matrix

(c) unit matrix (d) None of these

18) If any two rows and coloumns of a determinant are identical, the value of the determinant is

(a) 1 (b) 0 (c) -1 (d) unaltered

19) If there is only one column in a matrix, it is called

(a) Row matrix (b) column matrix

(c) square matrix (d) rectangular

20) Addition of matrices is

(a) not commutative (b) commutative

(c) not associative (d) distributive

21) A square matrix A is said to be non-singular if

(a) $|A| \neq 0$ (b) $|A| = 0$ (c) $A = 0$ (d) None of these

22) The value of x if $\begin{vmatrix} 1 & x \\ 5 & 3 \end{vmatrix} = 0$ is

(a) $\frac{5}{3}$ (b) $\frac{3}{5}$ (c) 0 (d) None of these

23) If $\begin{vmatrix} 4 & 8 \\ -9 & 4 \end{vmatrix} = 88$, then the value of $\begin{vmatrix} 8 & 4 \\ 4 & -9 \end{vmatrix}$ is

(a) -88 (b) 88 (c) 80 (d) None of these

24) The value of $\begin{vmatrix} 3 & 2 \\ 3 & 2 \end{vmatrix}$ is

(a) 0 (b) -1 (c) 1 (d) None of these

25) If $\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = -2$, then the value of $\begin{vmatrix} 2 & 6 \\ 2 & 4 \end{vmatrix}$ is

(a) -2 (b) 2 (c) -4 (d) None of these

26) If $(A+B)(A-B) = A^2 - B^2$ and A and B are square matrices then

(a) $(AB)^T = AB$ (b) $AB = BA$

(c) $(A+B)T = B^T + A^T$ (d) None of these

27) $\begin{pmatrix} 10 & 10 \\ 10 & 10 \end{pmatrix}$ is a

(a) Rectangular matrix (b) Scalar matrix

(c) Identity matrix (d) None of these

28) $\begin{pmatrix} 1 \\ 2 \\ 6 \\ 7 \end{pmatrix}$ is a

(a) Square matrix (b) Row matrix

(c) Scalar matrix (d) Column matrix

29) If $A = I$, then A^2

(a) I^2 (b) I (c) 0 (d) None of these

30) If $A = (1 \ 2 \ 3)$ and $B = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ then the order of AB is

(a) 1 x 1 (b) 1 x 3 (c) 3 x 1 (d) 3 x 3

PERMUTATIONS

This topic deals with the new Mathematical idea of counting without doing actual counting. That is without listing out particular cases it is possible to assess the number of cases under certain given conditions.

Permutations refer to different arrangement of things from a given lot taken one or more at a time. For example, Permutations made out of a set of three elements {a,b,c}

(i) One at a time: {a}, {b}, {c} ...... 3 ways

(ii) Two at a time: {a,b}, {b,a},{b,c}, {c,b}, {a,c}, {c,a} ...... 6 ways

(iii) Three at a time: {a,b,c}, {a,c,b}, {b,c,a}, {b,a,c}, {c,a,b}, {c,b,a} ......6 ways

2.2.1 Fundamental rules of counting

There are two fundamental rules of counting based on the simple principles of multiplication and addition, the former when events occur independently one after another and latter when either of the events can occur simultaneously. Some times we have to combine the two depending on the nature of the problem.

2.2.2 Fundamental principle of counting

Let us consider an example from our day-to-day life. Sekar was allotted a roll number for his examination. But he forgot his number. What all he remembered was that it was a two digit odd number.

The possible numbers are listed as follows:

11	21	31	41	51	61	71	81	91
13	23	33	43	53	63	73	83	93
15	25	35	45	55	65	75	85	95
17	27	37	47	57	67	77	87	97
19	29	39	49	59	69	79	89	99

So the total number of possible two digit odd numbers = 9x5 = 45

Let us see whether there is any other method to find the total number of two digit odd numbers. Now the digit in the unit place can be any one of the five digits 1,3,5,7,9. This is because our number is an odd number. The digit in the ten's place can be any one of the nine digits 1,2,3,4,5,6,7,8,9

Thus there are five ways to fill up the unit place and nine ways to fill up the ten's place. So the total number of two digit odd numbers = 9x5 = 45. This example illustrates the following principle.

(i) Multiplication principle

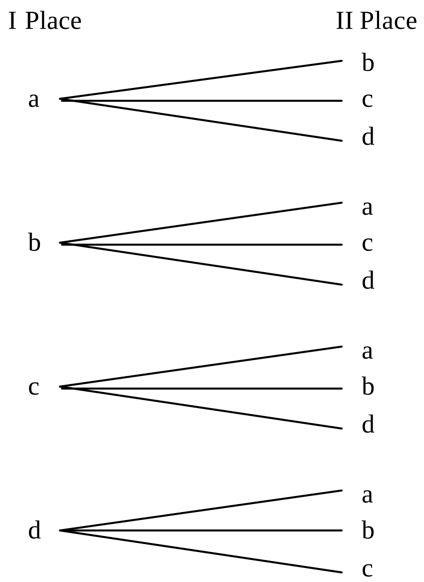
If one operation can be performed in "m" different ways and another operation can be performed in "n" different ways then the two operations together can be performed in 'm x n' different ways. This principle is known as *multiplication principle* of counting.

(ii) Addition Principle

If one operation can be performed in *m* ways and another operation can be performed in *n* ways, then any one of the two operations can be performed in m+n ways. This principle known as *addition principle* of counting.

Further consider the set {a,b,c,d}

From the above set we have to select two elements and we have to arrange them as follows.



The possible arrangements are

(a,b), (a,c), (a,d)
(b,a), (b,c), (b,d)
(c,a), (c,b), (c,d)
(d,a), (d,b), (d,c)

The total number of arrangements are 4 x 3 = 12

In the above arrangement, the pair (a,b) is different from the pair (b,a) and so on. There are 12 possible ways of arranging the letters a,b,c,d taking two at a time.

i.e Selecting and arranging '2' from '4' can be done in 12 ways. In otherwords number of permutations of 'four' things taken 'two' at a time is 4x3 = 12

In general $^{n}p_{r}$ denotes the number of permutations of 'n' things taken 'r' at a time.

['n' and 'r' are positive integers and $r \leq n$]

2.2.3 To find the value of $^{n}p_{r}$:

$^{n}p_{r}$ means selecting and arranging 'r' things from 'n' things which is the same as filling 'r' places using 'n' things which can be done as follows.

The first place can be filled by using anyone of 'n' things in 'n' ways

The second place can be filled by using any one of the remaining (n-1) things in (n-1) ways.

So the first and the second places together can be filled in n(n-1) ways.

The third place can be filled in (n-2) ways by using the remaining (n-2) things.

So the first, second and the third places together can be filled in n(n-1) (n-2) ways.

In general 'r' places can be filled in n(n-1)(n-2)....[n-(r-1)] ways.

So $^{n}p_{r}$ = n(n-1) (n-2)...(n-r+1). To simplify the above formula, we are going to introduce factorial notation.

2.2.4 Factorial notation:

The product of first 'n' natural numbers is called n- factorial denoted by n ! or $\underline{|n}$.

For example:

$$5! = 5 \times 4 \times 3 \times 2 \times 1$$
$$4! = 4 \times 3 \times 2 \times 1$$
$$\therefore 5! = 5 \times 4!$$
$$5! = 5 \times 4 \times 3!$$

In general, $n! = n(n-1)(n-2)...3.2.1$

$$\therefore n! = n\{(n-1)!\}$$
$$= n(n-1)(n-2)! \text{ and so on}$$

We have $^{n}p_{r} = n(n-1)(n-2)............(n-r+1)$

$$= \frac{n(n-1)(n-2)......(n-r+1)(n-r)!}{(n-r)!} = \frac{n!}{(n-r)!}$$

{multiplying and dividing by (n-r)!}

$$\boxed{\therefore\ ^{n}p_{r} = \frac{n!}{(n-r)!}}$$

Observation :

(i) $o! = 1$

(ii) $^{n}p_{o} = \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1$

(iii) $^{n}p_{1} = \frac{n!}{(n-1)!} = \frac{n(n-1)!}{(n-1)!} = n$

(iv) $^{n}p_{n} = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$

(ie. Selecting and arranging 'n' things from 'n' things can be done in n! ways).

(i.e 'n' things can be arranged among themselves in n! ways).

2.2.5 Permutations of repeated things:

If there are 'n' things of which 'm' are of one kind and the remaining (n-m) are of another kind, then the total number of distinct permutations of 'n' things

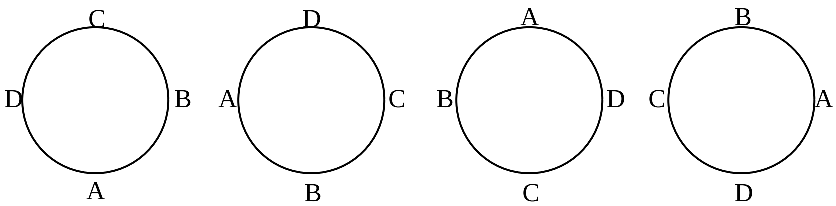
$$= \frac{n!}{m!(n-m)!}$$

If there are m_1 things of first kind, m_2 things of second kind and m_r things of r^{th} kind such that $m_1+m_2+.....+m_r = n$ then the total number of permutations of 'n' things

$$= \frac{n!}{m_1!m_2!......m_r!}$$

2.2.6 Circular Permutations:

We have seen permutations of 'n' things in a row. Now we consider the permutations of 'n' things in a circle. Consider four letters A,B,C,D. The four letters can be arranged in a row in 4! ways. Of the 4! arrangements, the arrangement ABCD, BCDA, CDAB, DABC are the same when represented along a circle.



So the number of permutations of '4' things along a circle is $\frac{4!}{4} = 3!$

In general, n things can be arranged among themselves in a circle in (n-1)! ways

Example 6

Find the value of (i) $^{10}p_1$**,** **(ii)** $^{7}p_4$**,** **(iii)** $^{11}p_0$

Solution:

i) $^{10}p_1 = 10$

ii) $^{7}p_4 = \frac{\lfloor 7}{\lfloor 7-4} = \frac{\lfloor 7}{\lfloor 3} = \frac{7 \times 6 \times 5 \times 4 \times 3!}{3!} = 7 \times 6 \times 5 \times 4 = 840$

iii) $^{11}p_0 = 1$

Example 7

There are 4 trains from Chennai to Madurai and back to Chennai. In how many ways can a person go from Chennai to Madurai and return in a different train?

Solution:

Number of ways of selecting a train from Chennai to Madurai from the four trains = $^{4}p_1$ = 4ways

Number of ways of selecting a train from Madurai to Chennai from the remaining 3 trains = $^{3}p_1$ = 3 ways

$\therefore$ Total number of ways of making the journey = 4 x 3 = 12ways

Example 8

There is a letter lock with 3 rings each marked with 4 letters and do not know the key word. How many maximum useless attempts may be made to open the lock?

Solution:

To open the lock :

The number of ways in which the first ring's position can be fixed using the four letters = $^{4}p_1$ = 4 ways

The number of ways in which the second ring's position can be fixed using the 4 letters = $^{4}p_1$ = 4 ways

The number of ways in which the third ring's position can be fixed using the 4 letters = $^4p_1$ = 4 ways

$\therefore$ Total number of attempts = 4 x 4 x 4 = 64 ways

Of these attempts, only one attempt will open the lock.

$\therefore$ Maximum number of useless attempts = 64 - 1 = 63

Example 9

How many number of 4 digits can be formed out of the digits 0,1,2,..........,9 if repetition of digits is not allowed.

Solution:

The number of ways in which the 1000's place can be filled (0 cannot be in the 1000's place) = 9ways

The number of ways in which the 100's place 10's place and the unit place filled using the remaining 9 digits (including zero) = $^9p_3$ = 504 ways

$\therefore$ Total number of 4 digit numbers formed = 9 x 504 = 4536

Example 10

Find the number of arrangements of 6 boys and 4 girls in a line so that no two girls sit together

Solution:

Six boys can be arranged among themselves in a line in 6! ways. After this arrangement we have to arrange the four girls in such a way that in between two girls there is atleast one boy. So the possible places to fill with the girls are as follows

☐ B ☐ B ☐ B ☐ B ☐ B ☐ B ☐

The four girls can be arranged in the boxes (7 places) which can be done in $^7p_4$ ways. So the total number of arrangements = 6! x $^7p_4$ = 720 x 7 x 6 x 5 x 4 = 604800

Example 11

A family of 4 brothers and 3 sisters are to be arranged in a row. In how many ways can they be seated if all the sisters sit together?

Solution:

Consider the 3 sisters as one unit. There are 4 brothers which is treated as 4 units. Now there are totally 5 units which can be arranged among themselves in 5! ways. After these arrangements the 3 sisters can be arranged among themselves in 3! ways.

$\therefore$ Total number of arrangement = 5! x 3! = 720

Example 12

Find the sum of all the numbers that can be formed with the digits 2, 3, 4, 5 taken all at a time.

Solution:

Number of 4 digit numbers that can be formed using the digits 2, 3, 4, 5 is ${}^4p_4$=4! = 24. Out of the 24 numbers the digit 2 appears in the unit place 6 times, the digit 3 appears in the unit place 6 times and so on. If we write all the 24 numbers and add, the sum of all the numbers in the unit place

= 6[2+3+4+5] = 6 x 14 = 84

Similarly the sum of all the numbers in the 10's place = 84
The sum of all the numbers in the 100's place = 84
and the sum of all the numbers in the 1000's place = 84
$\therefore$ sum of all the 4 digit numbers = 84x1000 + 84x100+84x10+84x1
= 84 (1000+100+10+1) = 84 x 1111
= 93324

Example 13

In how many ways can the letters of the word CONTAMINATION be arranged?

Solution:

The number of letters of word CONTAMINATION = 13
which can be arranged in 13! ways
Of these arrangements the letter O occurs 2 times
N occurs 3 times
T occurs 2 times
A occurs 2 times
and I occurs 2 times

$\therefore$ The total number of permutations = $\frac{13!}{2!\,3!\,2!\,2!\,2!}$

EXERCISE 2.2

1) If ${}^{n}p_5 = (42)\,{}^{n}p_3$, find n

2) If $6[{}^{n}p_3] = 7\,{}^{(n-1)}p_3$ find n

3) How many distinct words can be formed using all the letters of the word
i) ENTERTAINMENT ii) MATHEMATICS iii) MISSISSIPPI

4) How many even numbers of 4 digits can be formed out of the digits 1,2,3,....9 if repetition of digits is not allowed?

5) Find the sum of all numbers that can be formed with the digits 3,4,5,6,7 taken all at a time.

6) In how many ways can 7 boys and 4 girls can be arranged in a row so that
i) all the girls sit together ii) no two girls sit together?

7) In how many ways can the letters of the word STRANGE be arranged so that vowels may appear in the odd places.

8) In how many ways 5 gentlemen and 3 ladies can be arranged along a round table so that no two ladies are together?

9) Find the number of words that can be formed by considering all possible permutations of the letters of the word FATHER. How many of these words begin with F and end with R?

2.3 COMBINATIONS

Combination are selections ie. it inolves only the selection of the required number of things out of the total number of things. Thus in combination order does not matter.

For example, consider a set of three elements {a,b,c} and combination made out of the set with

i) One at a time: {a}, {b}, {c}
ii) Two at a time: {a,b}, {b,c}, {c,a}
iii) Three at a time: {a,b,c}

The number of comibnations of n things taken r, $(r \leq n)$ is denoted by ${}^{n}c_r$ or $\binom{n}{r}$

2.3.1 To derive the formula for ^nc_r:

Number of combinations of 'n' things taken 'r' at a time $= {}^nc_r$
Number of permutations of 'n' things taken 'r' at a time $= {}^np_r$
Number of ways 'r' things can be arranged among themselves $= r!$
Each combination having r things gives rise to r! permutations

$$\therefore \quad {}^np_r = ({}^nc_r)\, r!$$

$$\Rightarrow \quad \frac{n!}{(n-r)!} = ({}^nc_r)\, r!$$

$$\therefore \quad {}^nc_r = \frac{n!}{r!\,(n-r)!}$$

Observation:

(i) $\quad {}^nc_o = \dfrac{n!}{0!\,(n-0)!} = \dfrac{n!}{n!} = 1$

(ii) $\quad {}^nc_n = \dfrac{n!}{n!\,(n-n)!} = \dfrac{n!}{r!0!} = 1$

(iii) $\quad {}^nc_r = {}^nc_{n-r}$

(iv) $\quad$ If ${}^nc_x = {}^nc_y$ then $x = y$ or $x+y = n$

(v) $\quad {}^nc_r = \dfrac{{}^np_r}{r!}$

Example14

Evaluate $^8p_3$ and $^8c_3$

Solution:

$${}^8p_3 = \frac{8!}{(8-3)!} = \frac{8!}{5!} = \frac{8x7x6x5!}{5!} = 8 \text{ x } 7 \text{ x } 6 = 336$$

$${}^8c_3 = \frac{8!}{3!\,(8-3)!} = \frac{8!}{3!\ 5!} = \frac{8x7x6x5!}{3!5!} = \frac{8x7x6}{3x2x1} = 56$$

Example 15

Evaluate $^{10}c_8$

Solution:

$${}^{10}c_8 = {}^{10}c_2 = \frac{10x9}{2x1} = 45$$

Example 16

If $^nc_8 = {^nc_6}$, find $^nc_2$.

Solution:

$^nc_8 = {^nc_6}$ (given)

$\Rightarrow n = 8+6 = 14$

$\therefore \quad {^nc_2} = {^{14}c_2} = \dfrac{14 \times 13}{2 \times 1} = 91$

Example 17

If $\binom{100}{r} = \binom{100}{4r}$, find 'r'

Solution:

$^{100}c_r = {^{100}c_{4r}}$ (given)

$\Rightarrow \quad r + 4r = 100$

$\therefore \quad r = 20$

Example 18

Out of 7 consonants and 4 vowels, how many words can be made each containing 3 consonants and 2 vowels.

Solution:

Selecting 3 from 7 consonants can be done in $^7c_3$ ways

Selecting 2 from 4 vowels can be done in $^4c_2$ ways.

$\therefore$ Total number of words formed $= {^7c_3} \times {^4c_2}$

$= \dfrac{7 \times 6 \times 5}{3 \times 2 \times 1} \times \dfrac{4 \times 3}{2 \times 1}$

$\therefore = 35 \times 6 = 210$

Example 19

There are 13 persons in a party. If each of them shakes hands with each other, how many handshakes happen in the party?

Solution:

Selecting two persons from 13 persons can be done in $^{13}c_2$ ways.

$\therefore$ Total number of hand shakes $= {^{13}c_2} = \dfrac{13 \times 12}{2 \times 1} = 78$

Example 20

There are 10 points in a plane in which none of the 3 points are collinear. Find the number of lines that can be drawn using the 10 points.

Solution:

To draw a line we need atleast two points. Now selecting 2 from 10 can be done in $^{10}c_2$ ways

$\therefore$ number of lines drawn = $^{10}c_2 = \frac{10 \times 9}{2 \times 1} = 45$

Example 21

A question paper has two parts, part A and part B each with 10 questions. If the student has to choose 8 from part A and 5 from part B, in how many ways can he choose the questions?

Solution:

Number of questions in part A = 10.
Selecting 8 from part A can be done in $^{10}c_8$ ways = $^{10}c_2$

Number of questions in part B = 10
Selecting 5 from part B can be done in $^{10}c_5$ ways

$\therefore$ Total number of ways in which the questions can be selected

$= {}^{10}c_8 \times {}^{10}c_5 = 45 \times 252 = 11340$ ways

Example 22

A committee of seven students is formed selecting from 6 boys and 5 girls such that majority are from boys. How many different committees can be formed?

Solution:

Number of students in the committee = 7
Number of boys = 6
Number of girls = 5
The selection can be done as follows

Boy (6)	Girl (5)
6	1
5	2
4	3

ie. (6B and 1G) or (5B and 2G) or (4B and 3G)

The possible ways are $\binom{6}{6}\binom{5}{1}$ or $\binom{6}{5}\binom{5}{2}$ or $\binom{6}{4}\binom{5}{3}$

$\therefore$ The total number of different committees formed

$$= {}^6c_6 \times {}^5c_1 + {}^6c_5 \times {}^5c_2 + {}^6c_4 \times {}^5c_3$$

$$= 1 \times 5 + 6 \times 10 + 15 \times 10 = 215$$

2.3.2 Pascal's Triangle

For n = 0, 1, 2, 3, 4, 5 ... the details can be arranged in the form of a triangle known as Pascal's triangle.

$$\begin{array}{lccccccccccc}
n = 0 & & & & & & \binom{0}{0} & & & & & \\
n = 1 & & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
n = 2 & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
n = 3 & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
n = 4 & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
n = 5 & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}$$

Substituting the values we get

$$\begin{array}{lccccccccccc}
n = 0 & & & & & & 1 & & & & & \\
n = 1 & & & & & 1 & & 1 & & & & \\
n = 2 & & & & 1 & & 2 & & 1 & & & \\
n = 3 & & & 1 & & 3 & & 3 & & 1 & & \\
n = 4 & & 1 & & 4 & & 6 & & 4 & & 1 & \\
n = 5 & 1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

The conclusion arrived at from this triangle named after the French Mathematician Pascal is as follows. The value of any entry in any row is equal to sum of the values of the two entries in the preceding row on either side of it. Hence we get the result.

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

2.3.3 Using the formula for ^nc_r derive that $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$

Proof :

L.H.S. $= {}^nc_r + {}^nc_{r-1}$

$$= \frac{n!}{r!\,(n-r)!} + \frac{n!}{(r-1!)[n-(r-1)]!}$$

$$= \frac{n!}{r!\,(n-r)!} + \frac{n!}{(r-1!)(n-r+1)!}$$

$$= \frac{n!\,[n-r+1] + n!\,(r)}{r!\,(n+1-r)!}$$

$$= \frac{n!\,[n-r+1+r]}{r!\,(n-r+1)!} = \frac{n!(n+1)}{r!(n-r+1)!}$$

$$= \frac{(n+1)!}{r!(n-r+1)!} = \frac{(n+1)!}{r!(n+1-r)!}$$

$= {}^{n+1}c_r =$ R.H.S.

EXERCISE 2.3

1) Evaluate a) ${}^{10}c_6$ b) ${}^{15}c_{13}$

2) If ${}^{36}c_n = {}^{36}c_{n+4}$, find ‘n’.

3) ${}^{n+2}c_n = 45$, find n.

4) A candidate is required to answer 7 questions out of 12 questions which are divided into two groups each containing 6 questions. He is not permitted to attempt more than 5 questions from each group. In how many ways can he choose the 7 questions.

5) From a set of 9 ladies and 8 gentlemen a group of 5 is to be formed. In how many ways the group can be formed so that it contains majority of ladies

6) From a class of 15 students, 10 are to be chosen for an excursion party. There are 3 students who decide that either all of them will join or none of them will join. In how many ways can they be chosen.

7) Find the number of diagonals of a hexagon.

8) A cricket team of 11 players is to be chosen from 20 players including 6 bowlers and 3 wicket keepers. In how many different ways can a team be formed so that the team contains exactly 2 wicket keepers and atleast 4 bowlers.

2.4 MATHEMATICAL INDUCTION

Many mathematical theorems, formulae which cannot be easily derived by direct proof are sometimes proved by the indirect method known as mathematical induction. It consists of three steps.

(i) Actual verification of the theorem for n = 1

(ii) Assuming that the theorem is true for some positive integer k(k>1). We have to prove that the theorem is true for k+1 which is the integer next to k.

(iii) The conclusion is that the theorem is true for all natural numbers.

2.4.1 Principle of Mathematical Induction:

Let P(n) be the statement for $n \in N$. If P(1) is true and P(k+1) is also true whenever P(k) is true for k > 1 then P(n) is true for all natural numbers.

Example 23

Using the principle of Mathematical Induction prove that for all $n \in N$, $1+2+3+...n = \frac{n(n+1)}{2}$

Solution:

Let $P(n) = \frac{n(n+1)}{2}$

For L.H.S. n=1, p(1) = 1

For R.H.S $p(1) = \frac{1(1+1)}{2} = 1$

L.H.S = R.H.S for n = 1

$\therefore$ P(1) is true.

Now assume that P(k) is true

i.e. $1+2+3+......+k = \frac{k(k+1)}{2}$ is true.

To prove that p(k+1) is true

Now **p(k+1) = p(k) + t_{k+1}**

p(k+1) = 1+2+3+......k+k+1

$= p(k) + (k+1)$

$= \frac{k(k+1)}{2} + k+1$

$= (k+1)\left[\frac{k}{2}+1\right]$

$= \frac{(k+1)(k+2)}{2}$

=> p(k+1) is true whenever p(k) is true. But p(1) is true.

$\therefore$ p(n) is true for all $n \in N$.

Example 24

Show by principle of mathematical induction that $3^{2n} - 1$ is divisible by 8 for all $n \in N$.

Solution:

Let P(n) be the given statement

p(1) $= 3^2 - 1 = 9 - 1 = 8$ which is divisible by 8.

$\therefore$ p(1) is true.

Assume that p(k) is true

ie., $3^{2k}-1$ is divisible by 8.

To prove p(k+1) is true.

Now p(k+1) $= 3^{2(k+1)} - 1 = 3^{2k} \times 3^2 - 1$

$= 9\ 3^{2k} - 1$

$= 9(3^{2k}) - 9 + 8$

$= 9[3^{2k}-1] + 8$

Which is divisible by 8 as $3^{2k}-1$ is divisible by 8

So p(k+1) is true whenever p(k) is true. So by induction p(n) is true for all $n \in N$.

EXERCISE 2.4

By the principle of mathematical induction prove the following

1) $1+3+5+\ldots\ldots (2k-1) = k^2$

2) $4+8+12+\ldots\ldots .4n = 2n(n+1)$

3) $1.2 + 2.3 + 3.4 + \ldots\ldots n(n+1) = \frac{n(n+1)(n+2)}{3}$

4) $1^3 + 2^3 + \ldots\ldots n^3 = \frac{n^2(n+1)^2}{4}$

5) $1^2 + 2^2 + \ldots\ldots n^2 = \frac{n(n+1)(2n+1)}{6}$

6) $1+4+7+10+\ldots\ldots(3n-2) = \frac{n}{2}(3n-1)$

7) $2^{3n} - 1$ is divisible by 7.

2.4.2 Summation of Series

We have $1+2+3+\ldots\ldots+n = \Sigma n = \frac{n(n+1)}{2}$

$$1^2+2^2+\ldots\ldots+n^2 = \Sigma n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^3+2^3+\ldots\ldots+n^3 = \Sigma n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$$

Thus $\boxed{\mathbf{Sn} = \frac{n(n+1)}{2}}$

$$\boxed{\mathbf{Sn^2} = \frac{n(n+1)(2n+1)}{6}}$$

$$\boxed{\mathbf{Sn^3} = \left\{\frac{n(n+1)}{2}\right\}^2}$$

Using the above formula we are going to find the summation when the nth term of the sequence is given.

Example 25

Find the sum to n terms of the series whose nth term is n(n+1)(n+4)

Solution :

$t_n = n(n+1)(n+4)$

$= n^3 + 5n^2 + 4n$

$\therefore S_n = \Sigma t_n = \Sigma(n^3 + 5n^2 + 4n)$

$= \Sigma n^3 + 5\,\Sigma n^2 + 4\Sigma n$

$$= \left\{\frac{n(n+1)}{2}\right\}^2 + 5\left\{\frac{n(n+1)(2n+1)}{6}\right\} + 4\left\{\frac{n(n+1)}{2}\right\}$$

$$= \frac{n(n+1)}{12}\ [3n^2 + 23n + 34]$$

Example 26

Sum to n terms of the series $1^2.3 + 2^2.5 + 3^2.7 + ...$

Solution:

The n^{th} term is $n^2(2n+1) = 2n^3+n^2$

$$\therefore S_n = \Sigma(2n^3+n^2) = 2\Sigma n^3 + \Sigma n^2$$

$$= \frac{2n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n(n+1)}{2}\left[n(n+1) + \frac{2n+1}{3}\right]$$

$$= \frac{n(n+1)}{2}\left(\frac{3n^2+3n+2n+1}{3}\right)$$

$$= \frac{n(n+1)}{6}[3n^2 + 5n + 1]$$

Example 27

Sum the following series 2+5+10+17+......to n terms

Solution:

2+5+10+17+.......

= (1+1) + (1+4) + (1+9) + (1+16)+......

= (1+1+1+.......n terms) + ($1^2+2^2+.....n^2$)

$$= n+\frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n}{6}[6+2n^2+3n+1]$$

$$= \frac{n}{6}[2n^2+3n+7]$$

EXERCISE 2.5

Find the sum to n terms of the following series

1) 1.2.3 + 2.3.4 + 3.4.5 + ...

2) $1.2^2 + 2.3^2 + 3.4^2 + \ldots\ldots\ldots$

3) $2^2 + 4^2 + 6^2 + \ldots\ldots\ldots (2n)^2$

4) 2.5 + 5.8 + 8.11 + ...

5) $1^2 + 3^2 + 5^2 + \ldots\ldots\ldots$

6) 1 + (1+2) + (1+2+3) + ...

2.5 BINOMIAL THEOREM

2.5.1 Theorem

If n is a natural number,

$(x+a)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1}\, a + {}^nC_2\, x^{n-2}\, a^2 + \ldots\ldots + {}^nC_r\, x^{n-r}\, a^r + \ldots.\ {}^nC_n\ a^n$

Proof:

We shall prove the theorem by the principle of Mathematical Induction

Let P(n) denote the statement :

$(x+a)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1}\, a + {}^nC_2\, x^{n-2}\, a^2 + \ldots\ldots$
$+ {}^nC_{r-1}\, x^{n+1-r}\, a^{r-1} + {}^nC_r\, x^{n-r}\, a^r + \ldots\ldots + {}^nC_n\ a^n$

Let n = 1, Then LHS of P(1) = x + a

RHS of P(1) = 1 . x + 1 . a = x + a = L.H.S. of P (1)

$\therefore$ P (1) is true

Let us assume that the statement P (k) be true for $k \in N$

i.e. P(k) :

$(x+a)^k = {}^kC_0\, x^k + {}^kC_1\, x^{k-1}\, a + {}^kC_2\, x^{k-2}\, a^2 + \ldots\ldots$
$+ {}^kC_{r-1}\, x^{k+1-r}\, a^{r-1} + {}^kC_r\, x^{k-r}\, a^r + \ldots\ldots + {}^kC_k\, a^k$ .... (1)

is true

To prove P (k+1) is true

i.e., $(x+a)^{k+1} = {}^{k+1}C_0\, x^{k+1} + {}^{k+1}C_1\, x^k\, a$
$+ {}^{k+1}C_2\, x^{k-1}\, a^2 + \ldots + {}^{k+1}C_r\, x^{k+1-r}\, a^r + \ldots + \ldots + {}^{k+1}C_{k+1}\, a^{k+1}$ is true.

$(x+a)^{k+1} = (x+a)\,(x+a)^k$
$= (x + a)\,[{}^kC_0\, x^k + {}^kC_1\, x^{k-1}\, a + {}^kC_2\, x^{k-2}\, a^2 + \ldots + {}^kC_{r-1}\, x^{k+1-r}\, a^{r-1}$
$+ {}^kC_r\, x^{k-r}\, a^r + \ldots. + {}^kC_k\, a^k]$ using (1)

$= {}^kC_0\, x^{k+1} + {}^kC_1\, x^k a + {}^kC_2\, x^{k-1} a^2 + \dots + {}^kC_r\, x^{k+1-r} a^r + \dots + {}^kC_k\, x\, a^k$

$\qquad + {}^kC_0\, x^k a + {}^kC_1\, x^{k-1} a + \dots {}^kC_{r-1}\, x^{k+1-r} a^r + \dots + {}^kC_k\, a^{k+1}$

$= {}^kC_0\, x^{k+1} + ({}^kC_1 + {}^kC_0)\, x^k a + ({}^kC_2 + {}^kC_1)\, x^{k-1} a^2 + \dots\dots$

$\qquad \dots\dots + ({}^kC_r + {}^kC_{r-1})\; x^{k+1-r} a^r + \dots + {}^kC_k\, a^{k+1}$

But ${}^kC_r + {}^kC_{r-1} = {}^{k+1}C_r$

Put r = 1, 2, ... etc.

${}^kC_1 + {}^kC_0 = {}^{k+1}C_1,\ {}^kC_2 + {}^kC_1 = {}^{k+1}C_2 \dots\dots$

${}^kC_0 = 1 = {}^{k+1}C_0\ ;\ {}^kC_k = 1 = {}^{k+1}C_{k+1}$

$\therefore (x+a)^{k+1} = {}^{k+1}C_0\, x^{k+1} + {}^{k+1}C_1\, x^k a + {}^{k+2}C_2\, x^{k-1} a^2 + \dots\dots$

$\qquad + {}^{k+1}C_r\, x^{k+1-r} a^r + \dots\dots + {}^{k+1}C_{k+1}\, a^{k+1}$

Thus if P (k) is true, then P (k +1) is also true.

$\therefore$ By the principle of mathematical induction P(n) is true for $n \in N$.

Thus the Binomial Theorem is proved for $n \in N$.

Observations:

(i) The expansion of $(x+a)^n$ has (n+1) terms.

(ii) The general term is given by $t_{r+1} = nC_r\, x^{n-r} a^r$.

(iii) In $(x+a)^n$, the power of 'x' decreases while the power of 'a' increases such that the sum of the indices in each term is equal to n.

(iv) The coefficients of terms equidistant from the beginning and end are equal.

(v) The expansion of $(x+a)^n$ has (n+1) terms Let n+1 = N.
a) when N is odd the middle term is $t_{\frac{N+1}{2}}$
b) when N is even the middle terms are $t_{\frac{N}{2}}$ and $t_{\frac{N}{2}+1}$

(vi) Binomial cooefficients can also be represented by C_0, C_1, C_2, etc.

2.5.2 Binomial coefficients and their properties

$(1+x)^n = C_0 + C_1 x + C_2\, x^2 + C_3\, x^3 + \dots + C_n x^n \quad \dots\dots\dots\dots(1)$

Put $x = 1$ in (1) we get

$2^n = C_0 + C_1 + C_2 + \dots\dots + C_n$

Put $x = -1$ in (1) we get

$0 = C_0 - C_1 + C_2 - C_3 + \dots + (-1)^n C_n$

$\Rightarrow C_0 + C_2 + C_4 + \dots\dots = C_1 + C_3 + \dots$

=> sum of the coefficients of even terms $= \dfrac{2^n}{2} = 2^{n-1}$

sum of the coefficients of odd terms $= 2^{n-1}$

Example 28

Expand $(x+\frac{1}{x})^4$

Solution :

$(x+\frac{1}{x})^4 = 4C_0\, x^4 + 4C_1\, x^3(\frac{1}{x}) + 4C_2\, x^2(\frac{1}{x})^2 + 4C_3\, x(\frac{1}{x})^3 + 4C_4(\frac{1}{x})^4$

$= x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4}$

Example 29

Expand $(x+3y)^4$

Solution :

$(x+3y)^4 = 4C_0\, x^4 + 4C_1\, x^3(3y) + 4C_2\, x^2\,(3y)^2 + 4C_3\, x(3y)^3 + 4C_4(3y)^4$

$= x^4 + 4x^3(3y) + 6x^2(9y^2) + 4x(27y^3) + 81y^4$

$= x^4 + 12x^3y + 54x^2y^2 + 108xy^3 + 81y^4$

Example 30

Find the 5th term of $(2x-3y)^7$

Solution :

$t_{r+1} = 7C_r(2x)^{7-r}\,(-3y)^r$

$\therefore \quad t_5 = t_{4+1} = 7C_4\,(2x)^{7-4}\,(-3y)^4$

$= 7C_3\,(2x)^3\,(3y)^4$

$= \frac{7 \times 6 \times 5}{3 \times 2 \times 1}\,(8x^3)\,(81y^4)$

$= (35)\,(8x^3)\,(81y^4) = 22680x^3y^4$

Example 31

Find the middle term(s) in the expansion of $(x-\frac{2}{x})^{11}$

Solution :

n = 11

$\therefore$ n+1 = 12 = N = even number

So middle terms = $t_{\frac{N}{2}}$ and $t_{(\frac{N}{2}+1)}$

ie., t_6 and t_7

(i) Now $t_6 = t_{5+1}$ $= 11C_5\, x^{11-5} (-\frac{2}{x})^5$

$= 11C_5\, x^6 \frac{(-2)^5}{x^5}$

$= -11C_5 \frac{x^6 2^5}{x^5}$

$= -11C_5\, 2^5\, x = (-11C_5)(32x)$

(ii) $t_7 = t_{6+1} = 11C_6\, (x)^{11-6} (-\frac{2}{x})^6$

$= 11C_6\, x^5 \frac{(-2)^6}{x^6}$

$= 11C_6 \frac{x^5 2^6}{x^6}$

$= 11C_6 \left(\frac{64}{x}\right)$

Example 32

Find the coefficient of x^{10} in the expansion of $(2x^2 - \frac{3}{x})^{11}$

Solution :

General term $= t_{r+1} = 11C_r (2x^2)^{11-r} (-\frac{3}{x})^r$

$= 11C_r\, 2^{11-r} (x^2)^{11-r} \frac{(-3)^r}{x^r}$

$= 11C_r\, 2^{11-r}\, x^{22-2r} (-3)^r\, x^{-r}$

$= 11C_r\, 2^{11-r} (-3)^r\, x^{22-3r}$

To find the coefficient of x^{10}, the index of x must be equated to 10.

$\Rightarrow 22-3r = 10$

$22-10 = 3r$

$\therefore r = 4$

So coefficient of x^{10} is $11C_4\, 2^{11-4} (-3)^4 = 11C_4 (2^7) (3^4)$

Example 33

Find the term independent of x in the expansion of $(\frac{4x^2}{3} - \frac{3}{2x})^9$

Solution :

General term $= t_{r+1} = 9C_r(\frac{4x^2}{3})^{9-r} (\frac{-3}{2x})^r$

$= 9C_r \frac{4^{9-r}}{3^{9-r}}$ x $\frac{(-3)^r}{2^r}$ x $(x^2)^{9-r} \frac{1}{x^r}$

$= 9C_r \frac{4^{9-r}}{3^{9-r}}$ x $\frac{(-3)^r}{2^r} x^{18-2r} x^{-r}$

$= 9C_r \frac{4^{9-r}}{3^{9-r}} \frac{(-3)^r}{2^r} x^{18-3r}$

The term independent of x = constant term = coefficient of x^0

$\therefore$ To find the term independent of x

The power of x must be equated to zero

=> 18-3r = 0

$\therefore$ r = 6

So the term independent of x is $9C_6 \frac{4^{9-6}}{3^{9-6}} \frac{(-3)^6}{2^6}$

$= 9C_3 \frac{4^3}{3^3} \frac{(3)^6}{(2)^6}$

$= \frac{9x8x7}{3x2x1}$ x $\frac{64}{3^3}$ x $\frac{3^6}{64}$

$= (84)(3^3) = 84x27 = 2268$

EXERCISE 2.6

1) Find the middle term(s) in the expansion of $(x- \frac{2}{x})^{11}$

2) Find the coefficient of x^{-8} in the expansion of $(x-- \frac{2}{x})^{20}$

3) Find the term independent of x in the expansion of $(x^2 - \frac{4}{x^3})^{10}$

4) Find the 8th term in the expansion of $(2x + \frac{1}{y})^9$

5) Find the middle term in the expansion of $(3x - \frac{x^3}{6})^9$

6) Find the term independent of x in the expansion of $(2x^2 + \frac{1}{x})^{12}$

7) Show that the middle term in the expansion of $(1+x)^{2n}$ is

$$\frac{1.3.5 \ldots (2n-1)\ 2^n . x^n}{n!}$$

8) Show that the middle term in the expansion of $(x + \frac{1}{2x})^{2n}$ is $\frac{1.3.5 \ldots (2n-1)}{n!}$

EXERCISE 2.7

Choose the correct answer

1) If n! = 24 then n is

(a) 4 (b) 3 (c) 4! (d) 1

2) The value of 3! + 2! + 1! + 0! is

(a) 10 (b) 6 (c) 7 (d) 9

3) The value of $\frac{1}{4!} + \frac{1}{3!}$ is

(a) $\frac{5}{20}$ (b) $\frac{5}{24}$ (c) $\frac{7}{12}$ (d) $\frac{1}{7}$

4) The total number of ways of analysing 6 persons around a table is

(a) 6 (b) 5 (c) 6! d) 5!

5) The value of x(x-1) (x-2)! is

(a) x! (b) (x-1)! (c) (x-2)! (d) (x+1)!

6) 2 persons can occupy 7 places in ____ ways

(a) 42 (b) 14 (c) 21 (d) 7

7) The value of ${}^8p_3$ is

(a) 8 x 7 x 6 (b) $\frac{8 \times 7 \times 6}{3 \times 2 \times 1}$ (c) 8 x 7 (d) 3 x 2 1

8) The value of ${}^8C_0$ is

(a) 8 (b) 1 (c) 7 (d) 0

9) The value of ${}^{10}C_9$ is

(a) 9 (b) 1 (c) ${}^{10}C_1$ (d) 0

10) Number of lines that can be drawn using 5 points in which none of 3 points are collinear is

(a) 10 (b) 20 (c) 5 (d) 1

11) If $\binom{5}{x} + \binom{5}{4} = \binom{6}{5}$ then x is

(a) 5 (b) 4 (c) 6 (d) 0

12) If ${}^{10}c_r = {}^{10}c_{4r}$ then r is

(a) 2 (b) 4 (c) 10 (d) 1

13) Sum of all the binomial coefficients is

(a) 2^n (b) b^n (c) 2n (d) n

14) The last term in $(x+1)^n$ is

(a) x^n (b) b^n (c) n (d) 1

15) The number of terms in $(2x+5)^7$ is

(a) 2 (b) 7 (c) 8 (d) 14

16) The middle term in $(x+a)^8$ is

(a) t_4 (b) t_5 (c) t_6 (d) t_3

17) The general term in $(x+a)^n$ is denoted by

(a) t_n (b) t_r (c) t_{r-1} (d) t_{r+1}